Common Coupled Coincidence Fixed Point Results in Generalized Partially Symmetric Metric Space

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***Abstract* — In this paper, we obtain common coupled coincidence fixed point theorem using E.A property for mappings in generalized partially symmetric metric space. This theorem extend, generalize and improve many results in the literature. An example is given.**

***Keywords*— Partially ordered set, G-metric space, symmetric G-metric space, coupled coincidence point, common coupled fixed point, mixed monotone property.**

1. Introduction

In 1992, B.C. Dhage introduced a new class of generalized metric space called D-metric spaces. Following this, Dhage developed topological structures in such spaces, generalized metric spaces and presented fixed point results. In 2004, Z. Mustafa and B. Sims [18] demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect. In 2006, they [19] introduced a valid generalized metric space structure called G-metric spaces. Z. Mustafa et al. [19] proved some fixed point theorems for mappings satisfying various contraction conditions in G-metric spaces. Abbas et al. [3] proved the study of common fixed point theorems in G-metric spaces. There are various results in G-metric spaces by many authors [4,8,9,10,11,12,13,14,16,17,20,22,23].

In the establishment of fixed point theorems in Partially ordered complete metric spaces with a contraction condition, Partial ordering relation is maintained between the points. Saadati et al. [24] proved some fixed point theorems in partially ordered *G*-metric spaces. In 2006, the notion of a coupled fixed point was introduced and studied by Guo and Lakshmikantham and Bhaskar. Abbas et al. [12] introduced the concepts of α and α*-compatible mappings. Abbas et al. [2] proved unique coupled fixed point using the concepts of α and α*-compatible mappings. Hassen Aydi et al. [5,6,7] Proved coupled coincidence and coupled common fixed point theorems for a mixed g-monotone mapping satisfying nonlinear contractions in partially ordered G-metric spaces. In 2012, Z. Kadelburg, Hemant Kumar Nahine and S. Radenovic [15] obtained improved version of some common coupled fixed point theorems for mappings in partially ordered symmetric *G*-metric spaces. Stojan Radenovic [21] used a method of reducing coupled coincidence point results in partially ordered symmetric *G*-metric spaces to the respective results for mappings with one variable. M. Aamri and D.El. Moutawakil [1] proved several common fixed point theorems for self-mappings satisfying a new property E.A. which generalize the notion of non-compatible maps in the setting of a symmetric space. The aim of this paper is to prove common coupled coincidence fixed point in partially symmetric *G*-metric space for mappings with E.A property.

2. Preliminaries

Following are the definitions and results concerning *G*-metric spaces.

Definition 2.1

Let $\mathcal{X}$ be a nonempty set and let $g : \mathcal{X}\times\mathcal{X}\times\mathcal{X}\rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $g(x,y,z)=0$ if $x=y=z$;

(G2) $0<g(x,x,y)$ for all $x,y\in\mathcal{X}$ with $x\neq y$;

(G3) $g(x,x,y)\leq g(x,y,z)$, for all $x,y,z\in\mathcal{X}$ with $z\neq y$;

(G4) $g(x,y,z)=g(x,z,y)=g(y,z,x)=\dots$, (symmetry in all three variables);

(G5) $g(x,y,z)\leq g(x,a,a)+g(a,y,z)$, for all $x,y,z,a\in\mathcal{X}$ (rectangle inequality).

Then the function g is called a *G*-metric on $\mathcal{X}$ and the pair $(\mathcal{X}, g)$ is called a *G*-metric space.

Definition 2.2

Let $(\mathcal{X}, g)$ be a *G*-metric space and let $\{x_n\}$ be a sequence of points in $\mathcal{X}$.

(1) A point $x\in\mathcal{X}$ is said to be the limit of a sequence $\{x_n\}$ if $\lim_{n,m\to\infty} g(x,x_n,x_m)=0$, and one says that the sequence $\{x_n\}$ is *g*-convergence to *x*.

(2) The sequence $\{x_n\}$ is said to be a g-Cauchy sequence if, for every $\varepsilon > 0$, there is a positive integer N such that $g(x_n, x_m, x_l) < \varepsilon$, for all $n,m,l \geq N$; that is, if $g(x_n, x_m, x_l) \to 0$, as $n,m,l \to \infty$.

(3) $(\mathcal{X}, g)$ is said to be g-complete (or a complete G-metric space) if every g-Cauchy sequence in $(\mathcal{X}, g)$ is a g-convergent in $\mathcal{X}$.

Definition 2.3

A G-metric space $(\mathcal{X}, g)$ is called symmetric if

$$g(x, y, y) = g(y, x, x)$$

holds for all $x, y \in \mathcal{X}$.

The following are some examples of G-metric spaces.

Example 2.1

(1) Let $(\mathcal{X}, d)$ be an ordinary metric space. Define g_s by

$$g_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in \mathcal{X}$. Then it is clear that $(\mathcal{X}, g_s)$ is a symmetric G-metric space.

(2) Let $\mathcal{X} = \{a, b\}$. Define

$$g(a,a,a) = g(b,b,b) = 0, \quad g(a,a,b) = 1, \quad g(a,b,b) = 2,$$

and extend g to $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ by using the symmetry in the variables. Then it is clear that $(\mathcal{X}, g)$ is an (asymmetric) G-metric space.

Remark 1

If $(\mathcal{X}, g)$ is a G-metric space, then

$$d_g(x, y) = g(x, y, y) + g(x, x, y)$$

define a standard metric on $\mathcal{X}$. If the G-metric g is symmetric, this reduces to $d_g(x, y) = 2g(x, y, y)$. It has to be noted that in some cases contraction conditions given in g-metric can be reformulated and used in this standard metric, but there are a lot of situations where it is not possible.

Definition 2.4

Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $f : \mathcal{X}^2 \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{X}$.

(1) f is said to have h-mixed monotone property if the following two conditions are satisfied:

$(\forall x_1, x_2, y \in \mathcal{X}) hx_1 \preceq hx_2 \Rightarrow f(x_1, y) \preceq f(x_2, y)$,

$(\forall x, y_1, y_2 \in \mathcal{X}) hy_1 \preceq hy_2 \Rightarrow f(x, y_1) \succeq f(x, y_2)$.

If $h = i_{\mathcal{X}}$ (the identity map), we say that f has the mixed monotone property.

(2) A point $(x, y) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled coincidence point of f and h if $f(x, y) = hx$ and $f(y, x) = hy$, and their common coupled fixed point if $f(x, y) = hx = x$ and $f(y, x) = hy = y$.

If $\mathcal{X}$ is a nonempty set, then the triple $(\mathcal{X}, g, \preceq)$ will be called an ordered G-metric space if:

(i) $(\mathcal{X}, g)$ is a G-metric space, and

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Definition 2.5

Let A and B be two self-mappings of a semi metric space (X, d). Then A and B satisfy the property E.A if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 2.6

Let $(\mathcal{X}, g)$ be a symmetric G-metric space. Let two mappings $h : \mathcal{X} \to \mathcal{X}$ and $f : \mathcal{X}^2 \to \mathcal{X}$ satisfy the property E.A if there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} g(f(x_n, y_n), a, a) = \lim_{n\to\infty} g(hx_n, a, a) = 0$.

Definition 2.7

An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called a couple fixed point for the mapping $f : \mathcal{X}^2 \to \mathcal{X}$ if $f(x, y) = x$ and $f(x, y) = y$.

Definition 2.8

An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called a coupled coincidence point of the mappings $f : \mathcal{X}^2 \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{X}$ if $f(x, y) = hx$ and $f(y, x) = hy$.

Definition 2.8

Let *X* be a non-empty set. We say that the mappings $h : \mathcal{X} \to \mathcal{X}$ and $f : \mathcal{X}^2 \to \mathcal{X}$ are commutative if $hf(x, y) = f(hx, hy)$.

Definition 2.9

The mappings $h : \mathcal{X} \to \mathcal{X}$ and $f : \mathcal{X}^2 \to \mathcal{X}$ are called weakly compatible if $hf(x, y) = f(hx, hy)$ whenever $hx = f(x, y)$ and $hy = f(y, x)$.

3. Main Results

Theorem 3.1

Let $(\mathcal{X}, g)$ be a symmetric *G*-metric space. Let two mappings $h : \mathcal{X} \to \mathcal{X}$ and $f : \mathcal{X}^2 \to \mathcal{X}$ satisfy the following conditions.

(1) *h* and *f* are weakly compatible.

(2) *h* and *f* satisfy (E.A) property.

(3) $f(\mathcal{X}^2) \subset h\mathcal{X}$ and $h\mathcal{X}$ is a complete subset of $\mathcal{X}$.

(4) For all $x, y, u, v, s, t \in \mathcal{X}$, the following condition holds:

$$g(f(x, y), f(u, v), f(s, t)) \le \max\{g(hx, hu, hs), g(hy, hv, ht)\}$$

Then *h* and *f* have unique common coupled coincidence fixed point.

Proof

Since *h* and *f* satisfy (E.A) property, there exists a sequence $\{x_n\}$, $\{y_n\}$ in $\mathcal{X}$ such that $\underset{n\to\infty}{Lt}\, g(f(x_n, y_n), a, a) = \underset{n\to\infty}{Lt}\, g(hx_n, a, a) = 0$ for some $a \in \mathcal{X}$.

$\underset{n\to\infty}{Lt}\, g(f(y_n, x_n), b, b) = \underset{n\to\infty}{Lt}\, g(hy_n, b, b) = 0$ for some $b \in \mathcal{X}$.

Since $\mathcal{X}$ is symmetric *G*-metric space, we have

$$\underset{n\to\infty}{Lt}\, g(hx_n, a, a) = \underset{n\to\infty}{Lt}\, g(a, a, hx_n) = 0$$

$$\underset{n\to\infty}{Lt}\, g(hy_n, b, b) = \underset{n\to\infty}{Lt}\, g(b, b, hy_n) = 0$$

By (G5), we have

$$g(f(x_n, y_n), a, hx_n) \le g(f(x_n, y_n), a, a) + g(a, a, hx_n)$$

$$\underset{n\to\infty}{Lt}\, g(f(x_n, y_n), a, hx_n) = 0$$

So we have $\underset{n\to\infty}{Lt}\, f(x_n, y_n) = \underset{n\to\infty}{Lt}\, hx_n = a$ s

$$\underset{n\to\infty}{Lt}\, f(y_n, x_n) = \underset{n\to\infty}{Lt}\, hy_n = b$$

Suppose $h\mathcal{X}$ is a complete subspace of $\mathcal{X}$. Then a = *hx* for some $x \in \mathcal{X}$.

and *b* = *hy* for some $y \in \mathcal{X}$

We claim that $f(x, y) = hx$; $f(y, x) = hy$.

Indeed by (4), we have

$$g(f(x_n, y_n), f(x, y), f(x, y)) \le \max\{g(hx_n, hx, hx), g(hy_n, hy, hy)\}$$

Taking limit as $n \to \infty$, we have

$$\underset{n\to\infty}{Lt}\, g(f(x_n, y_n), f(x, y), f(x, y)) = 0$$

So we have $f(x, y) = hx$; $f(y, x) = hy$

Therefore, $(x, y) \in \mathcal{X}^2$ is a coupled coincidence point of *f* and *h*. The weak compatibility of *f* and *h* implies that $f(hx, hy) = hf(x, y)$ and then $f(f(x, y), f(y, x)) = f(hx, hy) = hf(x, y) = h(hx)$;

$f(f(y,x), f(x,y)) = f(hy,hx) = hf(y,x) = h(hy)$.

Let us show that $(f(x,y), f(y,x))$ is a common coupled coincidence fixed point of *f* and *h*.

Suppose $f(f(x,y), f(y,x)) \neq f(x,y)$

In view of (4), it follows

$g(f(x,y), f(x,y), f(f(x,y), f(y,x)))$

$\leq \max\{g(hx,hx, f(x,y)), g(hy,hy, f(y,x))\}$

$= 0$

which is a contradiction.

Therefore, $f(x,y) = f(f(x,y), f(y,x)) = hf(x,y)$ and $f(x,y)$ is a common coupled coincidence fixed point of *f* and *h*. The proof is similar when $f(\mathcal{X}^2)$ is assumed to be a complete subspace of $\mathcal{X}$ since $f(\mathcal{X}^2) \subset h\mathcal{X}$.

If $f(x,y) = hx = x$ and $f(c,d) = hc = c$ and $x \neq c; y \neq d$, then (4) gives

$g(x,c,c) = g(f(x,y), f(c,d), f(c,d))$

$\leq \max\{g(hx,hc,hc), g(hy,hd,hd)\}$

$= g(x,c,c)$

which is a contradiction.

So $x = c$; $y = d$ which implies that

$f(x,y) = hx = x$; $f(y,x) = hy = y$. Then (x, y) is a unique common coupled coincidence fixed point of *f* and *h*.

Theorem 3.2

Let $(\mathcal{X}, g, \preceq)$ be a partially ordered symmetric G-metric space. Let $f : \mathcal{X}^2 \to \mathcal{X}$ $h : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(1) *f* has the mixed *h*-monotone property.

(2) $f(\mathcal{X}^2) \subset h\mathcal{X}$ and $h\mathcal{X}$ is a complete subspace of $\mathcal{X}$.

(3) *h* and *f* satisfy (E.A) property.

(4) *f* and *h* are weakly compatible.

(5) Assume $g(f(x,y), f(u,v), f(s,t)) \leq \max\{g(x,u,s), g(y,v,t)\}$ for all $x, u, s, y, v, t \in \mathcal{X}$ for which $hx \leq hu \leq hs \wedge hy \geq hv \geq ht$ or $hx \geq hu \geq hs \wedge hy \leq hv \leq ht$.

(6) There exist $x_0, y_0 \in \mathcal{X}$ such that $hx_0 \leq f(x_0,y_0) \wedge hy_0 \geq f(y_0,x_0)$ or $hx_0 \geq f(x_0,y_0) \wedge hy_0 \leq f(y_0,x_0)$.

Then *f* and *h* have unique common coupled coincidence fixed point.

Proof

By (6), let $x_0, y_0 \in X$ such that $hx_0 \leq f(x_0,y_0) \wedge hy_0 \geq f(y_0,x_0)$ or $hx_0 \geq f(x_0,y_0) \wedge hy_0 \leq f(y_0,x_0)$. Since $f(\mathcal{X}^2) \subset h\mathcal{X}$, we can choose $x_1, y_1 \in X$ such that $hx_1 = f(x_0,y_0)$ and $hy_1 = f(y_0,x_0)$. So $hx_0 \leq hx_1$ or $hy_0 \leq hy_1$.

By (1), we have $hx_0 \leq hx_1 \Rightarrow f(x_0,y) \leq f(x_1,y)$; $hy_0 \leq hy_1 \Rightarrow f(x,y_1) \leq f(x,y_0)$.

Again since $f(\mathcal{X}^2) \subset h\mathcal{X}$, we can choose $x_2, y_2 \in X$ such that $hx_1 \leq f(x_1,y_1) \wedge hy_1 \geq f(y_1,x_1)$ or $hx_1 \geq f(x_1,y_1) \wedge hy_1 \leq f(y_1,x_1)$. So $hx_2 = f(x_1,y_1)$ and $hy_2 = f(y_1,x_1)$. Therefore $hx_0 \leq hx_1$ or $hy_0 \leq hy_1$. Then by (1), we have $hx_1 \leq hx_2 \Rightarrow f(x_1,y) \leq f(x_2,y)$; $hy_1 \leq hy_2 \Rightarrow f(x,y_2) \leq f(x,y_1)$.

Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $hx_n = f(x_{n-1},y_{n-1}) \leq hx_{n+1} = f(x_n,y_n)$; $hy_n = f(y_{n-1},x_{n-1}) \leq hy_{n+1} = f(y_n,x_n)$.

By (3), $\underset{n\to\infty}{Lt}\, g(f(x_n,y_n),a,a) = \underset{n\to\infty}{Lt}\, g(hx_n,a,a) = 0$ for some $a \in \mathcal{X}$.

$\underset{n\to\infty}{Lt}\, g(f(y_n,x_n),b,b) = \underset{n\to\infty}{Lt}\, g(hy_n,b,b) = 0$ for some $b \in \mathcal{X}$.

Then by Theorem 3.1, Then (x, y) is a unique common coupled coincidence fixed point of *f* and *h*.

Example 3.3

Let $X =$ ▯ be equipped with standard order ≤ and G-metric given as $g(x,y,z) = \max\{|x-y|,|x-z|,|y-z|\}$. Then $(\mathcal{X}, g, \preceq)$ is a partially ordered symmetric G-metric space. Define $f : \mathcal{X}^2 \to \mathcal{X}$ as $f(x,y) = \frac{x-2y}{4}$ and $h : \mathcal{X} \to \mathcal{X}$ as $hx = \frac{x}{2}$. It is clear that f has the mixed h-monotone property. Also it is obvious that $f(\mathcal{X}^2) \subset h\mathcal{X}$ and $h\mathcal{X}$ is a complete subspace of $\mathcal{X}$.

As in the Theorem 3.2, all the conditions are satisfied.

Then (0, 0) is the unique common coupled fixed point of f and h.

4. Conclusion

In this work, we proved common coupled coincidence fixed point for mappings with E.A property in partially symmetric G-metric space. We gave an example for our results. Our results improved and extended various results existing in the literature.

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