Integrability and Similarity Analysis of deformed Kundu-Eckhaus Equation

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Abstract— In this paper, we investigate the integrability properties of deformed Kundu-Eckhaus equation. By applying Lie Classical method, we find Lie point symmetries of deformed Kundu-Eckhaus equation and using the obtained Lie point symmetries we derive its reduction equations.

Keywords— Lax pair, Lie point symmetries, Integrability, deformed Kundu-Eckhaus equation.

I. INTRODUCTION

The study of nonlinear evolution equations (NEEs) is very important, because it describes many real time situations in optical communication, engineering, etc [10]. In nonlinear optics, the optical pulse propagation plays vital role in telecommunication [10, 11, 19]. The well known nonlinear Schrodinger equation (NLSE) describes the optical soliton in the optical fiber. The nonlinear Schrodinger equation is given by

\[ iq_t + 2|q|^2q + q_{xx} = 0, \]  

(1.1)

where \( q(x,t) \) describes the complex function of \( x \) and \( t \), and the subscripts \( x \) and \( t \) denotes the partial derivatives with respect to \( x \) and \( t \). It is known that the NLSE (1.1) admits some remarkable mathematical structures such as the existence of Lax Pairs, multi-soliton solutions, infinitely many generalized symmetries, bi-Hamiltonian to multi-Hamiltonian representations, etc, [13, 16]. This leads to the integrability of NLSE (1.1).

In many situations, optical propagation demands some higher order nonlinearity. Kundu [11] shown that under a nonlinear transformation \( q \rightarrow Q = q e^{\lambda} \) with arbitrary gauge function \( \lambda \), the NLSE (1.1) is transformed to a nonlinear equation with higher order nonlinearity,

\[ iQ_t + Q_{xx} + 2|Q|^2Q - (\theta_t + \theta^2 - i\theta_x)Q + 2i\theta Q = 0, \]  

(1.2)

by the choice of gauge function \( \theta = 2x |q|^2 \) and \( \theta_x = 2ix(q_q - q_x) \) of the NLSE (1.1) to the derivative of the gauge function yields the Kundu-Eckhaus equation

\[ iq_t + 2|q|^2q + q_{xx} - 4ix(|q|^2)q + 4x^2 |q|^4 q = 0. \]  

(1.3)

The above equation (1.3) was derived by Kundu and Eckhaus separately in 1980’s with higher order nonlinearity, which arises in many situations, namely, communication, electromagnetic theory, etc. Many researchers [2, 7, 8, 13, 15, 16, 20, 22, 23, 24, 25] have dedicated their research work identifying the mathematical characteristics of NEEs, namely, Lax Pairs, multi-soliton solutions, infinitely many generalized symmetries, etc, and in particular for the Kundu-Eckhaus equation (1.3).

It is important to determine the complete integrability of nonholonomic deformation equations (by introducing the forcing terms in the NEEs along with the differential constraints in the form of a nonlinear differential equation involving spatial derivatives of the forcing terms) of NEEs by the characterization of the admission of rich mathematical structures mentioned above. Recently, many authors [1, 12, 14, 21, 22, 23, 24] contribute their research work towards the identification of nonholonomic deformation of NEEs which preserves the integrability properties of original NEEs.

In this paper, we consider the deformed Kundu-Eckhaus equation given by

\[ iq_t + 2|q|^2q + q_{xx} - 4ix(|q|^2)q + 4x^2 |q|^4 q = g - 2xag, \]  

(1.4a)

with the differential constraints

\[ g_x = 2ix |q|^2 g - 2iaq, \]  

(1.4b)

\[ a_x = i|q|^2 - iq^g. \]  

(1.4c)

where \( * \) denotes the complex conjugate of the function. \( a(x,t) \) is a real function and \( g(x,t) \) is a complex function. The rest of the paper is as follows: In section 2, we construct a Lax Pair for the deformed Kundu-Eckhaus equation by using Ablowitz, Kaup, Newell, and Segur (AKNS) procedure. In section 3, we derive the conservation laws of deformed Kundu-Eckhaus equation. In section 4, we discuss the group theoretical aspects of deformed Kundu-Eckhaus equation. Finally, we summarize our results.
II. LAX REPRESENTATION

One of the characterization of complete integrability of the NEE involving (1+1) independent variables in the sense of Lax is admitting the Lax representation. The main purpose of the Lax representation is to study the associated linear eigenvalue problem. Consider a linear eigenvalue problem

\[ \Phi_1 = L\Phi, \quad \Phi_2 = M\Phi, \]  

where \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)^T \). \( L, M \) are \( n \times n \) matrices. The compatibility condition, that is, \( \Phi_{\mu} = \Phi_{\nu} \), gives

\[ L_1 - M_1 + [L, M] = 0 \quad \text{or} \quad L_1 - M_1 + LM - ML = 0 \]  

which is known as the Lax equation and the matrices \( L \) and \( M \) are known as the Lax matrices (Lax pair). Ablowitz, Kaup, Newell, and Segur (AKNS) [2, 3, 15] systematically formulated the procedure to determine the explicit form of the Lax Matrices by expanding the matrix elements in terms of the spectral parameter \( \lambda \). which is known as the AKNS procedure.

By using this AKNS procedure, we obtain the Lax matrices \( L \) and \( M \) for the deformed Kundu-Eckhaus equation (1.4) and the explicit form are given by

\[ L = \lambda L_0 + L_1, \]
\[ M = 2\lambda^2 M_0 + \lambda M_1 + \lambda \frac{1}{\lambda} M_1, \]

where \( \lambda \) is a spectral parameter and \( L, M \) satisfies the Lax equation (2.2) with

\[ L_0 = M_0 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad L_1 = \begin{bmatrix} ik |q|^2 & q \\ -q^* & -i |q|^2 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 2q \\ -2q^* & 0 \end{bmatrix}, M_3 = \begin{bmatrix} \frac{ia}{2} & -\frac{g}{2} \\ \frac{g}{2} & \frac{ia}{2} \end{bmatrix}, \]

\[ M_2 = \begin{bmatrix} i |q|^2 + 4i\kappa^2 |q|^4 + \kappa(q q^* - q^* q) + i\kappa a \\ -2\kappa q^* |q|^2 + i\kappa a \end{bmatrix}, \]

then the compatibility condition (2.2) yields the deformed Kundu-Eckhaus equation (1.4).

III. CONSERVATION LAWS

In this section, we derive the conservation laws of deformed Kundu-Eckhaus equation (1.4) by using the Lax matrices obtained in the previous section.

First, we start with the spectral problem

\[ \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \mu - iqr \kappa & q \\ r & -\mu + iqr \kappa \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \]  

and

\[ \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} A(x,t,\mu) & B(x,t,\mu) \\ C(x,t,\mu) & -A(x,t,\mu) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \]  

where \( r = -q^* \) and \( \mu = -i\lambda \), then the compatibility condition gives

\[ A_{\mu} = -iq_r r \kappa - iqr \kappa - rB + qC, \]
\[ B_{\mu} = -2i B q r + 2B \mu - 2qA + q_r, \]
\[ C_{\mu} = 2i C q r \kappa - 2C \mu + 2A r. \]  

By taking \( \Gamma_1 = \frac{\phi_1}{\phi_2} \) and \( \Gamma_2 = \frac{\phi_2}{\phi_1} \), then the equation (3.1) and (3.2) reduced to Ricatti equations

\[ \frac{\partial}{\partial t}(-iqr \kappa + q \Gamma_1) = \frac{\partial}{\partial x} (A + B \Gamma_1), \]
\[ \frac{\partial}{\partial t}(iqr \kappa + r \Gamma_2) = \frac{\partial}{\partial x} (-A + C \Gamma_2). \]  

Rewriting (3.4a) and (3.4c), we have
Taking the power series expansion of \( q \Gamma_i \), in terms of \( \frac{1}{\mu} \) as

\[
q \Gamma_i = \sum_{n=0}^{\infty} f_n \left( \frac{1}{\mu} \right)^n.
\]

Making use of (3.10) in (3.8) and equating the like powers of \( \frac{1}{\mu} \) yields the relation

\[
f_{n+1} = \frac{1}{2} \left[ (rq) \delta_{n,0} - \sum_{k=0}^{n} f_k f_{n-k} + 2iqr \kappa f_n - q \left( \frac{f_n}{q} \right)_x \right],
\]

where \( \delta_{ij} \) is the Kronecker delta.

From (3.5a), the conserved density \( \rho \) and conserved flux \( I \) are of the form

\[
\rho = -iqr \kappa + q \Gamma_i,
\]

\[
I = A + BI_i,
\]

Substituting the power series (3.10) and equating the like powers of \( \frac{1}{\mu} \), we obtain the conserved densities as

\[
\rho^{(n)} = Cf_n
\]

and the corresponding conserved flux \( I^{(n)} \) can be obtained from the entries of the Lax matrix \( M \) using \( r = -q^* \) and \( \mu = -i\lambda \).

The explicit form of first three conserved densities \( \rho^{(n)}, n = 1, 2, 3 \) and the corresponding conserved fluxes \( I^{(n)}, n = 1, 2, 3 \) using the relation (3.8) are given below

\[
\rho^{(1)} = \|q\|^2, \quad I^{(1)} = i(q^* q_x - qq_x^*) + 4\kappa |q|^3 + \alpha, \\
\rho^{(2)} = \frac{1}{2} |q|^2, \quad \rho^{(2)} = \frac{1}{2} |q|^2 - i |q|^3 - iqq^* + 8\kappa |q|^2 q_x^* + 8i\kappa |q|^3, \\
\rho^{(3)} = -gq_x^* - |q|^3 (2i |q|^2 q_x^* + 6i |q|^2 q_{xx}^* + 6i |q|^2 q_{xxx}^* + 6i^2 q_{xxxx}^*) + 12\kappa^2 |q|^4, \\
I^{(3)} = 4 |q|^3 q_{xxx}^* + 2 |q|^3 q_x^* + |q|^5 q_{xxx}^* + 2 |q|^3 q_{xxxx}^* + 12\kappa^2 |q|^4.
\]

Using the relation (3.11) in (3.14) and for each \( n \), one can obtain the conserved densities \( \rho^{(n)} \) and the corresponding conserved flux \( I^{(n)} \). It is know that, there is a one-one correspondence between conserved quantities and the generalized symmetries [17]. Thus, the existence of infinitely many generalised symmetries guarantees another characteristics of integrability of deformed Kundu-Eckhaus equation (1.4).

IV. SIMILARITY ANALYSIS

In nineteenth century, Sophus Lie introduced the concept of Lie group technique to analyse the differential equations. Many researchers studied the applications of Lie Group method to analyse the differential equations [4, 5, 6, 7, 8, 9, 17, 18]. The concepts include the idea of invariant equations under one or several parameter Lie Group of transformations which helps to find the invariant solution of the differential equations. The transformation reduces the number of independent variables and hence it reduces the partial differential equation (PDE) into an ordinary differential equation (ODE).
A. Lie point symmetry of (1.4)

Consider the deformed Kundu-Eckhaus equation (1.4) along with its complex conjugates as
\[
\begin{align*}
\text{i}q_t + 2 |q|^2 q + q_{xx} - 4i\kappa (|q|^2), q + 4\kappa^2 |q|^4 q = g - 2iqa, \\
-iq'' + 2 |q|^2 q' + q'' + 4i\kappa (|q|^2), q' + 4\kappa^2 |q|^4 q' = g' - 2iqa',
\end{align*}
\]
where \( g_x = 2i\kappa |q|^2 g - 2iaq \), \( g'_x = -2i\kappa |q|^2 g' + 2iaq' \), \( a_x = iqg' - iq'g \).

Assume that equation (4.1) is invariant under a one parameter \( \varepsilon \) continuous point transformations,
\[
\begin{align*}
\bar{x} &= x + \varepsilon \xi + O(\varepsilon^2), \\
\bar{t} &= t + \varepsilon \tau + O(\varepsilon^2), \\
\bar{q} &= q + \varepsilon \eta_1 + O(\varepsilon^2), \\
\bar{q}' &= q' + \varepsilon \eta_2 + O(\varepsilon^2), \\
\bar{g}_x &= g + \varepsilon \phi_1 + O(\varepsilon^2), \\
\bar{g}_x' &= g' + \varepsilon \phi_2 + O(\varepsilon^2), \\
\bar{a}_x &= a + \varepsilon \psi + O(\varepsilon^2),
\end{align*}
\]
where \( \xi, \tau, \eta_1, \eta_2, \phi_1, \phi_2 \) and \( \psi \) are functions of \( q(t), q'(t), g(t), a(t) \) provided any solution \( q(x,t), q'(x,t), g(x,t), g'(x,t), a(x,t) \) satisfy equation (4.1). Then the infinitesimal transformation is given by
\[
v = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial q} + \eta_2 \frac{\partial}{\partial q'} + \phi_1 \frac{\partial}{\partial g_x} + \phi_2 \frac{\partial}{\partial g_x'} + \psi \frac{\partial}{\partial a},
\]
(4.3)
The second order prolongation of the vector field \( v \) is
\[
Pr^{(2)}v = v + \eta_{xx} \frac{\partial}{\partial q_{xx}} + \eta_{xx}' \frac{\partial}{\partial q'_{xx}} + \eta_{x} \frac{\partial}{\partial q_{x}} + \eta_{x}' \frac{\partial}{\partial q'_{x}} + \eta_{xx} \frac{\partial}{\partial g_{xx}} + \eta_{xx}' \frac{\partial}{\partial g'_{xx}} + \eta_{x} \frac{\partial}{\partial g_{x}} + \eta_{x}' \frac{\partial}{\partial g'_{x}} + \phi_{xx} \frac{\partial}{\partial g_{xx}} + \phi_{xx}' \frac{\partial}{\partial g'_{xx}} + \phi_{x} \frac{\partial}{\partial g_{x}} + \phi_{x}' \frac{\partial}{\partial g'_{x}} + \psi \frac{\partial}{\partial a} + \psi \frac{\partial}{\partial a},
\]
(4.4)
with
\[
\eta_{xx} = \frac{\partial \eta_{x}}{\partial x}, \quad \eta_{xx}' = \frac{\partial \eta_{x}'}{\partial x}, \\
\phi_{xx} = \frac{\partial \phi_{x}}{\partial x}, \quad \phi_{xx}' = \frac{\partial \phi_{x}'}{\partial x}, \\
\psi_{x} = \frac{\partial \psi}{\partial x}, \quad \psi_{x}' = \frac{\partial \psi'}{\partial x},
\]
where \( \frac{\partial}{\partial x} \) denotes the total differentiation, \( k = 1,2; i,j = 1,2 \), \( x_i = x \) and \( x_j = t \).

The invariant equations of (4.1) are given by
\[
\begin{align*}
Pr^{(2)}v (iq_t + 2 |q|^2 q + q_{xx} - 4i\kappa (|q|^2), q + 4\kappa^2 |q|^4 q - g + 2iqa) &= 0, \\
Pr^{(2)}v (-iq'' + 2 |q|^2 q' + q'' - 4i\kappa (|q|^2), q' + 4\kappa^2 |q|^4 q' - g' + 2iqa') &= 0, \\
Pr^{(2)}v (g_x - 2i\kappa |q|^2 g - 2iaq) &= 0, \\
Pr^{(2)}v (g_x' + 2i\kappa |q|^2 g' - 2iaq') &= 0, \\
Pr^{(2)}v (a_x - iqg' + iq'g) &= 0.
\end{align*}
\]
(4.5)
The invariant equations lead to a system of PDEs and solving them, we get the following Lie point symmetries,
\[
\xi = \beta, \quad \tau = \gamma, \quad \eta_1 = i\alpha a, \eta_2 = -i\alpha a', \quad \phi_1 = i\alpha g, \phi_2 = -i\alpha g', \quad \psi = 0,
\]
(4.6)
where \( \alpha, \beta, \gamma \) are constants. Then the infinitesimal generator (vector field) \( v \) becomes
\[
v = \beta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial t} + i\alpha a \frac{\partial}{\partial q} - i\alpha a' \frac{\partial}{\partial q'} + i\alpha g \frac{\partial}{\partial g_x} - i\alpha g' \frac{\partial}{\partial g_x'},
\]
(4.7)
which yields
\[
v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = i\alpha g \frac{\partial}{\partial q} - i\alpha a' \frac{\partial}{\partial q'} + i\alpha g \frac{\partial}{\partial g_x} - i\alpha g' \frac{\partial}{\partial g_x'},
\]
These vector fields form three dimension abelian Lie algebra A.

B. Group Invariants Solution of (1.4)

The infinitesimal generators \( v_i, i = 1, 2, 3 \) of the Lie algebra \( A \) induces the group of transformations. Thus the elements of one parameter (\( \dot{\theta} \)) group of transformations are

\[
\begin{align*}
g_1 : (x, t, q, g, q', g') & \rightarrow (x + \dot{\theta} t, q, q', g, g'), \\
g_2 : (x, t, q, g, q', g') & \rightarrow (x, t + \dot{\theta} q, q, g, g'), \\
g_3 : (x, t, q, g, q', g') & \rightarrow (x, t, q e^{\dot{\theta} g'}, q e^{\dot{\theta} g'}, g e^{\dot{\theta} g'}, g e^{\dot{\theta} g'})
\end{align*}
\]  

(4.8)

(4.9)

(4.10)

Hence, if \( q = f_1(x, t), q' = f_2(x, t), g = f_3(x, t), g' = f_4(x, t), a = f_5(x, t) \) is a solution of deformed Kundu-Eckhaus equation (1.4), then the function obtained under the action of transformations \( g_i(s), i = 1, 2, 3 \),

\[
\begin{align*}
g_1(s) & : f_1(x, t) \rightarrow f_1(x \dot{\theta} t), \\
g_2(s) & : f_1(x, t) \rightarrow f_2(x, t \dot{\theta}), \\
g_3(s) & : f_1(x, t) \rightarrow f_3(x, t e^{\dot{\theta} a}), \\
g_4(s) & : f_1(x, t) \rightarrow f_4(x, t e^{-\dot{\theta} a}), \\
g_5(s) & : f_1(x, t) \rightarrow f_5(x, t)
\end{align*}
\]  

is also a solution of deformed Kundu-Eckhaus equation (1.4). This solution is called the invariant solution.

C. Similarity reduction of (1.4)

In this subsection, we briefly explain how to derive reduction equations for the deformed Kundu-Eckhaus equation (1.4).

First, we derive the similarity variable and the similarity transformations by solving the Lagrange characteristic equation associated with the above obtained Lie point symmetries and then we can find the solution of the invariant surface condition for the vector field \( v \) given in (4.7). The main role of the (new transformed variable) similarity variable is to reduce (4.1) into ODE.

The characteristic equation associated with derived infinitesimals read

\[
\frac{dx}{f_1(x, t)} = \frac{dt}{f_2(x, t)} = \frac{dq}{f_3(x, t)} = \frac{dq'}{f_4(x, t)} = \frac{dg}{f_5(x, t)} = \frac{da}{f_6(x, t)} = 0.
\]  

(4.18)

By solving the above characteristic equation, we obtain the similarity variable \( \zeta(x, t) = \gamma x - \beta t \) and the similarity transformations \( Q(\zeta), R(\zeta), G(\zeta), H(\zeta) \) and \( A(\zeta) \) as

\[
\begin{align*}
q(x, t) & = Q(\zeta) e^{i\omega t}, \\
q'(x, t) & = R(\zeta) e^{-i\omega t}, \\
g(x, t) & = G(\zeta) e^{i\omega t}, \\
g'(x, t) & = G(\zeta) e^{-i\omega t}, \\
a(x, t) & = A(\zeta),
\end{align*}
\]  

(4.19)

Substituting the similarity variable and transformations into (4.1), we obtain the ordinary differential equations(ODEs), which is the similarity reduction,

\[
\begin{align*}
i\beta Q' - 4Q + 2Q^2 R + r^2 Q'' - 4ik\gamma Q(QR + RQ') + 4k^2 Q^2 R^2 & = G - 2kAQ \\
-\beta R' - 4R + 2QR + r^2 R'' + 4ik\gamma R(QR + RQ') + 4k^2 R^2 R' & = H - 2kAR \\
\gamma G' = 2\beta QGQ - 2iAQ \\
\gamma H' = -2\beta QRH + 2iAR \\
\gamma A = iQH - iRG.
\end{align*}
\]  

(4.20a)

(4.20b)

(4.20c)

(4.20d)

(4.20e)

One can obtain the solution of the deformed Kundu-Eckhaus equation (1.4) by using the solutions of the above ODEs.

V. CONCLUSIONS

In this paper, we investigated the integrability properties of deformed Kundu-Eckhaus equation, namely, existence of Lax representation and conservation laws. Using AKNS procedure, we derived the Lax pair for the deformed Kundu-Eckhaus equation. We have shown that the deformed Kundu-Eckhaus equation admits infinitely many conservation laws. The existence of Lax representation and infinitely many conservation laws.
laws both guarantees its integrability. We also derived Lie point symmetries of deformed Kundu-Eckhaus equation, using the obtained point symmetries group invariant solutions are discussed. Finally, we derived the reduction equations for the deformed Kundu-Eckhaus equation.

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