THE NEW OPERATOR OF OPEN SETS AND GENERALIZED CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. In this paper, we used the notion of operator A^+ for defining a new class of set which will be called α-+-open, pre-+-open, β-+-open, besides we define the concepts of α-+-irresolute, α-+-continuous, pre-+-irresolute, pre-+-continuous, β-+-irresolute and β-+-continuous. We define the concepts of generalized α-+-closed sets, generalized pre-+-closed sets and generalized β-+-closed sets. Moreover, some of their properties are shown.

1. Introduction and Preliminaries

The concept of operator A^+ was introduced by Elez and Papaz [5], they defined an operator A^+ by $A^+ = \mathrm{Cl(A)} - \mathrm{A}$. Furthermore, the study of operator, closure operator or interior operator have been grown in several topics in general topology, see [[1], [2], [9], [10] and [11]]. Carlos Granados [8] introduced and studied semi-+-open sets in topological spaces. S. Ganesan [7] introduced and studied b-+-open sets in topological spaces. In this paper, we used the operator A^+ for defining a new class of open sets which will be called α-+-open set, pre-+-open set and β-+-open set.

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Throughout this paper (X, τ) and (Y, σ) (or X and Y) (or) (K, τ) and (L, σ) (or K and L) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), Cl(A), Int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

Definition 1.1. *A subset A of a space (X, τ) is called:*

(1) α*-open set* [11] *if* $A \subseteq Int(Cl(Int(A)))$.

(2) *semi-open set* [9] *if* $A \subseteq Cl(Int(A))$.

(3) *preopen set* [10] *if* $A \subseteq Int(Cl(A))$.

(4) β*-open set* [1] *(= semi-preopen set* [2]*) if* $A \subseteq Cl(Int(Cl(A)))$.

(5) *regular open set* [12] *if* $A = Int(Cl(A))$.

The complements of the above mentioned open sets are called their respective closed sets.

Definition 1.2. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is called*

(1) *semi-+-open* [8] *if* $A^+ \subseteq Cl(Int(A^+))$.

(2) *b-+-open* [7] *if* $A^+ \subseteq Int(Cl(A^+)) \cup Cl(Int(A^+))$.

The complements of the above mentioned open sets are called their respective closed sets.

Remark 1.3. [8] *A^+ does not induce a topological space, because $X^+ = Cl(X) - X = X - X = \emptyset$ and $\emptyset^+ = Cl(\emptyset)$ - $\emptyset = \emptyset \cap X = \emptyset$. We can see that X will never be in the topology.*

Remark 1.4. [8] *The operator $A^+ = Cl(A) - A = Cl(A) \cap A^c$, where A^c means the complement of the set A.*

Lemma 1.5. [8] *Let* $\{A_\delta : \delta \in \Delta\}$ *be a collection of elements of (X, τ), then* $\bigcup_{\delta\in\Delta} A_\delta^+ \subseteq (\bigcup_{\delta\in\Delta} A_\delta)^+$.

Theorem 1.6. [7] *Every pre-+-open set is b-+-open set but not conversely.*

Definition 1.7. [7] *A function* $f : (K, \tau) \to (L, \sigma)$ *is said to be b-+-continuous if* $f^{-1}(A)$ *is b-+-open set in (K, τ) for every open set A of (L, σ).*

Definition 1.8. [8] *Let (X, τ) be a topological space and* $A \subseteq X$*. Then, A is said to be a generalized semi-+-closed set or simply gs-+-closed set if* $Cl_{s+(A)} \subseteq U$*, whenever* $A \subseteq U$ *and U is a open set.*
The complement of a gs+-closed set is called gs-+-open set.
The collection of all gs-+-closed sets and gs-+-open sets are denoted by GS+ (X, τ) and GS+ (X, τ), respectively.

Proposition 1.9. [8] *Every semi-+-closed set is gs-+-closed set but not conversely.*

Definition 1.10. [8] *Let (X, τ) be a topological space and* $A \subseteq X$*. Then, A is said to be a regular generalized semi-+-closed set or simply rgs-+-closed set if* $Cl_{s+(A)} \subseteq U$*, whenever* $A \subseteq U$ *and U is a regular open set.*
The complement of a rgs+-closed set is called rgs-+-open set.
The collection of all rgs-+-closed sets and rgs-+-open sets are denoted by RGS+ (X, τ) and RGS+ (X, τ), respectively.

Proposition 1.11. [8] *Every closed set is rgs-+-closed set but not conversely.*

Proposition 1.12. [8] *Every semi-+-closed set is rgs-+-closed set but not conversely.*

Definition 1.13. *A function* $f : (K, \tau) \to (L, \sigma)$ *is said to be continuous if* $f^{-1}(A)$ *is closed set in (K, τ) for every closed set A of (L, σ).*

2. α-+-open sets and generalized α-+-closed sets

Definition 2.1. *Let A subset A of a space (X, τ) is said to be α-+-open if $A^+ \subseteq$ Int(Cl(Int(A^+))).*

The complement of α-+-open sets is called α-+-closed sets.

The collection of all α-+-open sets and α-+-closed sets are denoted by α+O(X, τ) and α+C(X, τ) respectively.

We denote the power set of X by P(X).

Theorem 2.2. *Every closed set is α-+-open.*

Proof. Let A be a closed set of (X, τ). Since A is a closed set, then Cl(A) = A and so A^+ = Cl(A) − A = A ∩ $A^c = \phi$. Now, $\phi \subseteq$ Int(Cl(Int(A^+))) = $Int(Cl(Int(\phi)))$ = ϕ. This shows that A is α-+-open. □

Remark 2.3. *The following example shows that open sets is independent of α-+-open sets.*

Example 2.4. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then α-+-open sets are $\emptyset$, X, $\{c\}$, $\{a, c\}$, $\{b, c\}$. Here $\{a\}$ is a open set but it is not a α-+-open set. Also it is clear that $\{c\}$ is a α-+-open set but it is not a open set.*

Remark 2.5. *The following example shows that the notion of α-+-open set and α-+-open set are independent.*

Example 2.6. *Let X and $\tau(X)$ as in the Example 2.4. Then α-open sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$. Here $\{a, b\}$ is a α-open set but it is not a α-+-open set. Also it is clear that $\{a, c\}$ is a α-+-open set but it is not a α-open set.*

Theorem 2.7. *The arbitrary union of α-+-open sets is a α-+-open set.*

Proof. Let $\{A_i : i \in I\}$ be a family of α-+-open sets. Then for each i, $A_i^+ \subseteq$ $\text{Int}(\text{Cl}(\text{Int}(A_i^+))$

$\bigcup_{i\in I} A_i^+ \subseteq \bigcup_{i\in I} \text{Int}(\text{Cl}(\text{Int}(A_i^+)))$

$\subseteq \text{Int}(\text{Cl}(\bigcup_{i\in I}(\text{Int}(A_i^+))))$

$\subseteq \text{Int}(\text{Cl}(\text{Int}(\bigcup_{i\in I}(A_i^+))))$. By the Lemma 1.5, we have that

$\subseteq \text{Int}(\text{Cl}(\text{Int}(\bigcup_{i\in I}(A_i)^+)))$.

Thus, $\bigcup_{i\in I} A_i^+$ is a α-+-open set. □

Remark 2.8. *The arbitrary intersection of α-+-closed sets is α-+-closed set.*

Definition 2.9. *Let (X, τ) be a topological space and $A \subset X$. An element $x \in A$ is said to be α-+-interior point of A if there exits a α-+-open set U such that $x \in U \subseteq A$. The set of all α-+-interior points of A is said to be α+-interior of A an it is denoted by $Int_{\alpha+(A)}$.*

Theorem 2.10. *Let (X, τ) be a topological space and $A \subset X$. Then, A is α-+-open if and only if $A = Int_{\alpha+(A)}$.*

Proof. Let A be a α-+-open set. Then, $A \subseteq A$ and this implies that $A \in \{U \mid U$ is α-+-open and $U \subset A\}$. Since union of this collection is in A. Therefore, $A = \text{Int}_{\alpha+(A)}$. Conversely, suppose that $A = \text{Int}_{\alpha+(A)}$. Hence, A is α-+-open. □

Definition 2.11. *Let $A \subset X$. Then $x \in X$ is α-+-adherent to A if $U \cap A \neq \emptyset$ for every α-+-open set U containing x. The set of all α-+-adherent points of A is said to be α-+-closure of A and it is denoted by $Cl_{\alpha+(A)}$.*

Theorem 2.12. *Let (X, τ) be a topological space and $A, B \subseteq X$. Then, the following statements hold:*

(1) $A \subseteq Cl_{\alpha+(A)}$.

(2) $Cl_{\alpha+(A)}$ *is the smallest* α*-+-closed set containing* A*, that is* $Cl_{\alpha+(A)} = \bigcap \{ W \mid W$ *is* α*-+-closed ans* $A \subseteq W\}$.

(3) A *is* α*-+-closed if and only if* $A = Cl_{\alpha+(A)}$.

(4) *If* $A \subseteq B$*, then* $Cl_{\alpha+(A)} \subseteq Cl_{\alpha+(B)}$.

(5) $Cl_{\alpha+(A)} \cup Cl_{\alpha+(A)} \subseteq Cl_{\alpha+(A\cup B)}$.

(6) $Cl_{\alpha+(A\cap B)} \subseteq Cl_{\alpha+(A)} \cap Cl_{\alpha+(B)}$.

Proof. (1) Let x $\in$ A and suppose that x $\notin$ $\mathrm{Cl}_{\alpha+(A)}$. Then, there exits α-+-open set V containing x such that V $\cap$ A = $\emptyset$ and this is a contradiction. Therefore, x $\in$ $\mathrm{Cl}_{\alpha+(A)}$.
(2) Let x $\in$ $\mathrm{Cl}_{\alpha+(A)}$. Then, V $\cap$ A $\neq \emptyset$ for every α-+-open set V containing x. Now, suppose the contrary, that x $\notin \bigcap \{$ W $\mid$ W is α-+-closed ans A $\subseteq$ W$\}$. Then, x $\notin$ W for some α-+-closed set W, so x $\in$ X $-$ W for some α-+-open set X $-$ W. So, (X$-$W) $\cap$ A = $\emptyset$ for some α-+-open set X $-$ W containing x and this is a contradiction. Therefore, x $\notin$ $\{$ W $\mid$ W is α-+-closed ans A $\subseteq$ W$\}$. Conversely, let y $\in$ x $\notin \bigcap\{$ W $\mid$ W is α-+-closed ans A $\subseteq$ W$\}$. Then, y $\in$W for all α-+-closed set W containing A. Now, suppose that y $\notin$ $\mathrm{Cl}_{\alpha+(A)}$. Then, there exits α-+-open set V containing y such that V $\cap$ A = $\emptyset$. Therefore, X $-$ V is α-+-closed set containing A and y $\notin$ X $-$ V and this is a contradiction. Therefore, y $\in$ $\mathrm{Cl}_{\alpha+(A)}$.
The proof of (3) and (4) are followed directly from the Definition 2.11. (5) and (6) are followed by applying part (4) of this Theorem. □

Theorem 2.13. *Let (X,* τ*) be a topological space and A, B* $\subseteq$ *X. Then, the following statements hold:*

(1) *If* $A \subseteq B$*, then* $Int_{\alpha+(A)} \subseteq Int_{\alpha+(B)}$.

(2) $Int_{\alpha+(A)} \cup Int_{\alpha+(A)} \subseteq Int_{\alpha+(A\cup B)}$.

(3) $Int_{\alpha+(A\cap B)} \subseteq Int_{\alpha+(A)} \cap Int_{\alpha+(B)}$.

Proof. The proof of (1) is followed directly from the Definition 2.9. (2) and (3) are followed by applying part (1) of this Theorem. □

Theorem 2.14. *Let (X, τ) be a topological space and A, $B \subseteq X$. Then, the following statements hold:*

(1) $Int_{\alpha+(X\setminus A)} = X \setminus Cl_{\alpha+(A)}$.

(2) $Cl_{\alpha+(X\setminus A)} = X \setminus Int_{\alpha+(A)}$.

(3) $X \setminus Cl_{\alpha+(X\setminus A)} = Int_{\alpha+(A)}$.

(4) $X \setminus Int_{\alpha+(X\setminus A)} = Cl_{\alpha+(A)}$.

(5) $x \in Int_{\alpha+(A)}$ *if and only if there exists a* α*-+-open set M such that* $x \in M \subseteq A$.

Proof. Omitted. □

Theorem 2.15. *Let A be a subset of a topological space (X, τ). Then,* $x \in Cl_{\alpha+(A)}$ *if and only if for every* α*-+-open subset M of X containing x,* $A \cap M \neq \emptyset$.

Proof. Let x $\in$ $\mathrm{Cl}_{\alpha+(A)}$ and suppose that M $\cap$ A = $\emptyset$ for some α-+-open set M which contains x. Then, (X $\setminus$ M) is α-+-closed and A $\subset$ (X $\setminus$ M), thus $\mathrm{Cl}_{\alpha+(A)} \subset$ (X $\setminus$ M). But this implies that x $\in$ (X $\setminus$ M), a contradiction. Thus, A $\cap$ M $\neq \emptyset$.
Conversely, let A $\subseteq$ X and x $\in$ X such that for each α-+-open set M_1 which contains x, $M_1 \cap \neq \emptyset$. If x $\notin$ $\mathrm{Cl}_{\alpha+(A)}$, there is a α-+-closed set F such that A $\subseteq$ F and x $\notin$ F. Then, (X $\setminus$ F) is a α-+-open set with x $\in$ (X $\setminus$ F), and thus (X $\setminus$ F) $\cap$ A $\neq \emptyset$, which is a contradiction. □

Theorem 2.16. *Let (X, τ) be a topological space* $M \subseteq X$*. Then M is* α*-+-open if and only if for each* $s \in M$*, there exists a* α*-+-open set D such that* $s \in D \subseteq M$.

Proof. It is obvious. □

Definition 2.17. *A function $f: (K, \tau) \to (L, \sigma)$ is said to be α-+-irresolute if $f^{-1}(A)$ is α-+-open set in (K, τ) for every α-+-open set A of (L, σ).*

Theorem 2.18. *Let $f: (K, \tau) \to (L, \sigma)$ be a function, then the following statements are equivalent:*

(1) *f is α-+-irresolute.*

(2) *$f(Cl_{\alpha+(A)}) \subseteq Cl_{\alpha+(f(A))}$ holds for every subset A of K.*

(3) *$f^{-1}(A)$ is α-+-closed set in K, for every α-+-closed subset A of L.*

Proof. (2) $\Rightarrow$ (3): Let A be a α-+-closed set in L, then $\mathrm{Cl}_{\alpha+(A)} = \mathrm{A}$. By using (2), we have $\mathrm{f}(\mathrm{Cl}_{\alpha+f}{}^{-1}(\mathrm{A})) \subseteq \mathrm{Cl}_{\alpha+(A)} = \mathrm{A}$. Thus, $(\mathrm{Cl}_{\alpha+f}{}^{-1}(\mathrm{A})) \subseteq \mathrm{f}^{-1}(\mathrm{A})$ and hence $\mathrm{f}^{-1}(\mathrm{A})$ is α-+-closed in K.

(3) $\Rightarrow$ (2): If $\mathrm{A} \subseteq \mathrm{K}$, then $\mathrm{Cl}_{\alpha+(f(A))}$ is α-+-closed in L and by (3) $\mathrm{f}^{-1}(\mathrm{Cl}_{\alpha+(f(A))})$ is α-+-closed in K. Furthermore, $\mathrm{A} \subseteq \mathrm{f}^{-1}(\mathrm{f}(\mathrm{A})) \subseteq \mathrm{f}^{-1}(\mathrm{Cl}_{\alpha+(f(A))})$. Thus, $\mathrm{Cl}_{\alpha+(A)} \subseteq \mathrm{f}^{-1}(\mathrm{Cl}_{\alpha+(f(A))})$, consequently, $\mathrm{f}(\mathrm{Cl}_{\alpha+(A)}) \subseteq \mathrm{f}(\mathrm{f}^{-1}(\mathrm{Cl}_{\alpha+(f(A))})) \subseteq \mathrm{Cl}_{\alpha+(f(A))}$.

(3) $\Leftrightarrow$ (1): Obvious. □

Definition 2.19. *A function $f: X \to Y$ is said to be α-+-continuous at a point $x \in X$ if for each open subset K of Y containing f(x), there exists a α-+-open subset L of X containing x such that $f(L) \subseteq K$. The function f is said to be α-+-continuous if it has this property at each $x \in X$.*

Theorem 2.20. *A function $f: X \to Y$ is α-+-continuous if and only if the inverse image of every open set in Y is α-+-open in X.*

Proof. Let f be α-+-continuous and K be any open set in Y. If $\mathrm{f}^{-1}(\mathrm{K}) = \emptyset$, then $\mathrm{f}^{-1}(\mathrm{K})$ is a α-+-open set in X but if $\mathrm{f}^{-1}(\mathrm{K}) \neq \emptyset$, then there exists $\mathrm{x} \in \mathrm{f}^{-1}(\mathrm{K})$ which implies $\mathrm{f}(\mathrm{x}) \in \mathrm{K}$. Since f is α-+-continuous, then there exists a α-+-open set L in X containing x such that f(L) $\subseteq$ K. This implies that x $\in$ L $\subseteq$ f^{-1}(K) and hence f^{-1}(K) is α-+-open.

Conversely, let K be any open set in Y containing f(x), then x $\in$ f^{-1}(K) and by hypothesis f^{-1}(K) is a α-+-open set in X containing x, so f(f^{-1}(K)) $\subseteq$ K. Thus, f is α-+-continuous. □

Theorem 2.21. *For a function $f : X \to Y$, the following statements are equivalent:*

(1) *f is α-+-continuous.*

(2) *f^{-1}(K) is a α-+-open set in X, for each open subset K of Y.*

(3) *f^{-1}(F) is a α-+-closed set in X, for each closed subset F of Y.*

(4) *f($Cl_{\alpha+(A)}$) $\subseteq$ Cl(f(A)), for each subset A of X.*

(5) *$Cl_{\alpha+(f}{}^{-1}$(B))$\subseteq$ f^{-1}(Cl(B)), for each subset B of Y.*

(6) *f^{-1}(Int(B)) $\subseteq$ $Int_{\alpha+(f}{}^{-1}$(B)), for each subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Directly from Theorem 2.20.

(2) $\Rightarrow$ (3): Let F be any closed subset of Y. Then, Y $\setminus$ F is a open subset of Y. By (2), f^{-1}(Y $\setminus$ F) = X $\setminus$ f^{-1}(F) is a α-+-open set in X and hence f^{-1}(F) is a α-+-closed set in X.

(3) $\Rightarrow$ (4): Let A be any subset of X. Then, f(A) $\subseteq$ Cl(f(A)) and Cl(f(A)) is a closed set in Y. Hence, A $\subseteq$ f^{-1}(Cl(f(A))). By (3), we have f^{-1}(Cl(f(A))) is a α-+-closed set in X. Therefore, $Cl_{\alpha+(A)}$ $\subseteq$ f^{-1}(Cl(f(A))). Hence, f($Cl_{\alpha+(A)}$) $\subseteq$ Cl(f(A)).

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, f^{-1}(B) is a subset of X. By (4), we have f($Cl_{\alpha+(f}{}^{-1}$(B)))$\subseteq$ Cl(f(f^{-1}(B))) $\subseteq$ Cl(B). Hence, $Cl_{\alpha+(f}{}^{-1}$(B))$\subseteq$ f^{-1}(Cl(B)).

(5) $\Leftrightarrow$ (6): Let B be any subset of Y. Then, apply (5) to Y $\setminus$ B we obtain $Cl_{\alpha+(f}{}^{-1}$(Y $\setminus$ B))$\subseteq$ f^{-1}(Cl(Y $\setminus$ B)) $\Leftrightarrow$ $Cl_{\alpha+(X}\setminus$ f^{-1}(B))$\subseteq$ f^{-1}(X $\setminus$ Int(B)) $\Leftrightarrow$ X $\setminus$ $Int_{\alpha+(f}{}^{-1}$(B))$\subseteq$ X $\setminus$ f^{-1}(int(B)) $\Leftrightarrow$ f^{-1}(Int(B)) $\subseteq$ $Int_{\alpha+(f}{}^{-1}$(B)). Thus, f^{-1}(Int(B))) $\subseteq$ $Int_{\alpha+(f}{}^{-1}$(B)).

(6) ⇒ (1): Let x ∈ X and K be any open subset of Y containing f(x). By (6), we have $f^{-1}(Int(K)) \subseteq Int_{\alpha+(f}{}^{-1}(K))$ implies that $f^{-1}(K) \subseteq Int_{\alpha+(f}{}^{-1}(K))$. Hence, $f^{-1}(K)$ is a α-+-open set in X which contains x and clearly $f(f^{-1}(K)) \subseteq K$. Thus, f is α-+-continuous. □

Definition 2.22. *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be a generalized α-+-closed set or simply gα-+-closed set if $Cl_{\alpha+(A)} \subseteq U$, whenever $A \subseteq U$ and U is a open set.*
The complement of a gα-+-closed set is called gα-+-open set.
The collection of all gα-+-closed sets and gα-+-open sets are denoted by gα+ (X, τ) and gα+ (X, τ), respectively.

Proposition 2.23. *Every α-+-closed set is gα-+-closed set.*

Proof. The proof is followed by the Definition 2.22. □

The following example shows that the converse of the above Proposition, it is not always true.

Example 2.24. *Let X and $\tau(X)$ as in the Example 2.4. Then α-+-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$; gα+-closed sets are power set of X. Here $\{a, c\}$ is a gα-+-closed set but it is not a α-+-closed set.*

Theorem 2.25. *Let A be a gα+-closed subset of X. Then, $Cl_{\alpha+(A)} - A$ does not contain any non-empty closed sets.*

Proof. Let F be a closed set of X such that $F \subseteq Cl_{\alpha+(A)} - A$. Since X − F is a open set, then $A \subseteq X - F$ and A is *gα*+-closed, it follows $Cl_{\alpha+(A)} \subseteq X - F$, in consequence F $\subseteq$ X $-$ $\mathrm{Cl}_{\alpha+(A)}$. This implies that F $\subseteq$ (X $-$ $\mathrm{Cl}_{\alpha+(A)}$) $\cap$ ($\mathrm{Cl}_{\alpha+(A)}$ $-$ A) $= \emptyset$, therefore F $= \emptyset$. □

Corollary 2.26. *Let A be a $g\alpha$-+-closed set. Then, A is α-+-closed if and only if $Cl_{\alpha+(A)} - A$ is a closed set.*

Proof. Let A be $g\alpha$-+-closed set. If A is α-+-closed, it has $\mathrm{Cl}_{\alpha+(A)} - \mathrm{A} = \emptyset$ which is a closed set.

Conversely, let $\mathrm{Cl}_{\alpha+(A)} - \mathrm{A}$ be a closed set. Then, by the Theorem 2.25, $\mathrm{Cl}_{\alpha+(A)} - \mathrm{A}$ does not contain any non-empty closed set and $\mathrm{Cl}_{\alpha+(A)}$ is a closed set of itself. Thus, $\mathrm{Cl}_{\alpha+(A)} - \mathrm{A} = \emptyset$. Therefore, $\mathrm{A} = \mathrm{Cl}_{\alpha+(A)}$, in consequence A is a α-+-closed set. □

Corollary 2.27. *Let A be an open set and $g\alpha$-+-closed set. Then, $A \cap J$ is $g\alpha$-+-closed set whenever α-+-closed set J of X.*

Proof. Since A is $g\alpha$-+-closed and open set, then $\mathrm{Cl}_{\alpha+(A)} \subseteq \mathrm{A}$ and so A is a α-+-closed. Therefore, A $\cap$ J is α-+-closed set of X and this implies that A $\cap$ J is $g\alpha$-+-closed set of X. □

Theorem 2.28. *Let (X, τ) be a topological space and A, B $\subseteq$ X. If A is a $g\alpha$-+-closed set and B is any set such that $A \subseteq B \subseteq Cl_{\alpha+(A)}$, then B is a $g\alpha$-+-closed set of X.*

Proof. Let B $\subseteq$ V where V is an open set of X. Since A is a $g\alpha$-+-closed set and A $\subseteq$ V , then $\mathrm{Cl}_{\alpha+(A)} \subseteq \mathrm{V}$ and so $\mathrm{Cl}_{\alpha+(A)} = \mathrm{Cl}_{\alpha+(B)}$. Therefore, $\mathrm{Cl}_{\alpha+(B)} \subseteq \mathrm{V}$ and hence B is a $g\alpha$-+-closed set of X. □

Theorem 2.29. *Let (X, τ) be a topological space and $A \subset X$. A is a $g\alpha$-+-open set if and only if $J \subseteq Int_{\alpha+(A)}$ whenever J closed set and $J \subseteq A$.*

Proof. Let A be a $g\alpha$-+-open set and let J $\subseteq$ A where J is a closed set. Then, X $-$ A is a $g\alpha$-+-closed set contained in the open set X $-$ J. Therefore, $\mathrm{Cl}_{\alpha+}$(X$-$A) $\subseteq$ X $-$ J and X $-$ $\mathrm{Int}_{\alpha+(A)}$ $\subseteq$ X $-$ J. In consequence, J $\subseteq$ $\mathrm{Int}_{\alpha+(A)}$.
Conversely, if A is a closed set with J $\subseteq$ $\mathrm{Int}_{\alpha+(A)}$ and J $\subseteq$ A, then X $-$ $\mathrm{Int}_{\alpha+(A)}$ $\subseteq$ X $-$ J. Therefore, $\mathrm{Cl}_{\alpha+}$(X$-$A) $\subseteq$ X $-$ J. Hence, X $-$ A is a $g\alpha$-+-closed set and A is a $g\alpha$-+-open set of X. □

Definition 2.30. *A function f : (K, τ) $\rightarrow$ (L, σ) is said to be $g\alpha$-+-continuous if f^{-1}(A) is $g\alpha$-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 2.31. *Every α-+-continuous is $g\alpha$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 2.23. □

Definition 2.32. *Let (X, τ) be a topological space and A $\subseteq$ X. Then, A is said to be a regular generalized α-+-closed set or simply $rg\alpha$-+-closed set if $Cl_{\alpha+(A)}$ $\subseteq$ U, whenever A $\subseteq$ U and U is a regular open set.*

The complement of a $rg\alpha$-+-closed set is called $rg\alpha$-+-open set.

The collection of all $rg\alpha$-+-closed sets and $rg\alpha$-+-open sets are denoted by $rg\alpha$+(X, τ) and $rg\alpha$+(X, τ),respectively.

Proposition 2.33. *Every closed set is $rg\alpha$-+-closed set.*

Proof. Let B be any closed set of X such that B $\subseteq$ V . where V is a regular open set. Since $\mathrm{Cl}_{\alpha+(B)}$ $\subseteq$ Cl(B) = B. Therefore, $\mathrm{Cl}_{\alpha+(B)}$ $\subseteq$ V . In consequence, B is a $rg\alpha$+-closed set. □
The following example shows that the converse of the above Theorem need not be true.

Example 2.34. *Let X and $\tau(X)$ as in the Example 2.24. Then $rg\alpha$-+-closed sets are power set of X. Here, $\{a, b\}$ is $rg\alpha$-+-closed set but it is not closed set.*

Proposition 2.35. *Every α-+-closed set is $rg\alpha$-+-closed set.*

Proof. Let B be any α-+-closed set of X such that V is any regular open set containing B. Since B is a α-+- closed set, then $\mathrm{Cl}_{\alpha+(B)} = $ B. Therefore, $\mathrm{Cl}_{\alpha+(B)} \subseteq V$. Hence, B is a $rg\alpha$-+-closed set. □

The following example shows that the converse of the above Theorem need not be true.

Example 2.36. *Let X and $\tau(X)$ as in the Example 2.34. Here, $\{a, c\}$ is $rg\alpha$-+-closed set but it is not α-+-closed set.*

Proposition 2.37. *Every $g\alpha$-+-closed set is $rg\alpha$-+-closed set but not conveserly.*

Proof. If A is a $g\alpha$-+-closed subset of (X, τ) and G is any regular open set containing A, since every regular open set is open, we have G $\supseteq \mathrm{Cl}_{\alpha+(A)}$. Hence A is $rg\alpha$-+-closed in (X, τ). □

Example 2.38. *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b, c\}\}$. Then $g\alpha$-+-closed sets are $\emptyset$, X, $\{a\}$, $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d, \{b, c, d\}$; $rg\alpha$-+-closed sets are power of X. Here $\{a, b, c\}$ is $rg\alpha$-+-closed set but it is not a $g\alpha$-+-closed set.*

Definition 2.39. *A function $f : (K, \tau) \to (L, \sigma)$ is said to be regular generalized α-+-continuous (berifly, $rg\alpha$-+-continuous)if $f^{-1}(A)$ is $rg\alpha$-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 2.40. *Every continuous is $rg\alpha$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 2.33. □

Theorem 2.41. *Every α-+-continuous set is $rg\alpha$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 2.35. □

Proposition 2.42. *Every $g\alpha$-+-continuous is $rg\alpha$-+-continuous but not conveserly.*

Proof. The proof follows from Proposition 2.37. □

Proposition 2.43. *Every gs-+-closed set is rgs-+-closed set but not conveserly.*

Proof. If A is a gs-+-closed subset of (X, τ) and G is any regular open set containing A, since every regular open set is open, we have G $\supseteq$ $\mathrm{Cl}_{s+(A)}$. Hence A is rgs-+-closed in (X, τ). □

Example 2.44. *Let X and τ as in the Example 2.38. Then gs-+-closed sets are $\emptyset$, X, $\{a\}$, $\{d\}$, $\{a,\ d\}$, $\{b,\ d\}$, $\{c,\ d\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d,\ \{b,\ c,\ d\}$; rgs-+-closed sets are power of X. Here $\{a,\ c\}$ is rgs-+-closed set but it is not a gs-+-closed set.*

Definition 2.45. *A function f : X $\to$ Y is said to be semi-+-continuous at a point x $\in$ X if for each open subset K of Y containing f(x), there exists a semi-+-open subset L of X containing x such that f(L) $\subseteq$ K. The function f is said to be semi-+-continuous if it has this property at each x $\in$ X.*

Definition 2.46. *A function f : (K, τ) $\to$ (L, σ) is called*

(1) *generalized semi-+-continuous (berifly, gs-+-continuous) if $f^{-1}(A)$ is gs-+-closed set in (K, τ) for every closed set A of (L, σ).*

(2) *regular generalized semi-+-continuous (berifly, rgs-+-continuous) if $f^{-1}(A)$ is rgs-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 2.47. *Every semi-+-continuous is gs-+-continuous but not conversely.*

Proof. The proof follows from Proposition 1.9. □

Theorem 2.48. *Every continuous is rgs-+-continuous but not conversely.*

Proof. The proof follows from Proposition 1.11. □

Theorem 2.49. *Every semi+-continuous is rgs-+-continuous but not conversely.*

Proof. The proof follows from Proposition 1.12. □

Theorem 2.50. *Every gs-+-continuous is rgs-+-continuous but not conveserly.*

Proof. The proof follows from Proposition 2.43. □

3. pre-+-open sets and generalized pre-+-closed sets

Definition 3.1. *Let A subset A of a space (X, τ) is said to be pre-+-open if $A^+ \subseteq$ Int(Cl(A^+)).*

The complement of pre-+-open sets is called pre-+-closed sets.

The collection of all pre-+-open sets and pre-+-closed sets are denoted by p+O(X, τ) and p+C(X, τ) respectively.

Theorem 3.2. *Every closed set is pre-+-open.*

Proof. Let A be a closed set of (X, τ). Since A is a closed set, then Cl(A) = A and so A^+ = Cl(A) − A = A ∩ A^c = ϕ. Now, $\phi \subseteq$ Int(Cl(A^+)) = $Int(Cl(\phi))$ = ϕ. This shows that A is pre-+-open. □

Remark 3.3. *The following example shows that open sets is independent of pre-+-open sets.*

Example 3.4. *Let X and τ as in the Example 2.38. Then pre-+-open sets are $\emptyset$, X, $\{d\}$, $\{b, c, d\}$. Here $\{a, b, c\}$ is a open set but it is not a pre-+-open set. Also it is clear that $\{d\}$ is a pre-+-open set but it is not a open set.*

Remark 3.5. *The following example shows that the notion of pre-open set and pre-+-open set are independent.*

Example 3.6. *Let X and $\tau(X)$ as in the Example 3.4. Then pre-open sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$. Here $\{a, b, c\}$ is a pre-open set but it is not a pre-+-open set. Also it is clear that $\{b, c, d\}$ is a pre-+-open set but it is not a pre-open set.*

Remark 3.7. *The union (intersection) of any two pre-+-open sets in not pre-+-open. For,*

Example 3.8. *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. Then pre-+-open sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. (i) Let $M = \{a\}$ and $N = \{b\}$ are pre-+-open sets, but $A \cup B = \{a, b\}$ is not pre-+-open set. (ii) Let $P = \{b, d\}$ and $Q = \{c, d\}$ are pre-+-open sets, but $A \cap B = \{d\}$ is not pre-+-open set.*

Proposition 3.9. *Every α-+-open is pre-+-open.*

Proof. Let A be α-+-open then A $\subseteq$ Int(Cl(Int(A$^+$)))$\subseteq$ Int(Cl(A$^+$)).

Thus A is pre-+-open. □

The converse of the above Theorem need not be true in general as it is shown below.

Example 3.10. *Let X and $\tau(X)$ as in the Example 3.8. Then α-+-open sets are $\emptyset$, X, $\{c, d\}$. Here $\{a, d\}$ is pre-+-open set but it is not α-+-open set.*

Definition 3.11. *Let (X, τ) be a topological space and $A \subset X$. An element $x \in A$ is said to be pre-+-interior point of A if there exits a pre-+-open set U such that $x \in U \subseteq A$. The set of all pre-+-interior points of A is said to be pre+-interior of A an it is denoted by $Int_{p+(A)}$.*

Theorem 3.12. *Let (X, τ) be a topological space and $A \subset X$. Then, A is pre-+-open if and only if $A = Int_{p+(A)}$.*

Proof. Let A be a pre-+-open set. Then, A ⊆ A and this implies that A ∈ {U | U is pre-+-open and U ⊂ A}. Since union of this collection is in A. Therefore, A = $\text{Int}_{p+(A)}$.

Conversely, suppose that A = $\text{Int}_{p+(A)}$. Hence, A is pre-+-open. □

Definition 3.13. *Let $A \subset X$. Then $x \in X$ is pre-+-adherent to A if $U \cap A \notin \emptyset$ for every pre-+-open set U containing x. The set of all pre-+-adherent points of A is said to be pre-+-closure of A and it is denoted by $Cl_{p+(A)}$.*

Theorem 3.14. *Let (X, τ) be a topological space and $A, B \subseteq X$. Then, the following statements hold:*

(1) $A \subseteq Cl_{p+(A)}$.

(2) *$Cl_{p+(A)}$ is the smallest pre-+-closed set containing A, that is $Cl_{p+(A)} = \bigcap \{ W \mid W$ is pre-+-closed ans $A \subseteq W\}$.*

(3) *A is pre-+-closed if and only if $A = Cl_{p+(A)}$.*

(4) *If $A \subseteq B$, then $Cl_{p+(A)} \subseteq Cl_{p+(B)}$.*

(5) $Cl_{p+(A)} \cup Cl_{p+(A)} \subseteq Cl_{p+(A\cup B)}$.

(6) $Cl_{p+(A\cap B)} \subseteq Cl_{p+(A)} \cap Cl_{p+(B)}$.

Proof. (1) Let x $\in$ A and suppose that x $\notin$ $\mathrm{Cl}_{p+(A)}$. Then, there exits pre-+-open set V containing x such that V $\cap$ A = $\emptyset$ and this is a contradiction. Therefore, x $\in$ $\mathrm{Cl}_{p+(A)}$.

(2) Let x $\in$ $\mathrm{Cl}_{p+(A)}$. Then, V $\cap$ A $\neq \emptyset$ for every pre-+-open set V containing x. Now, suppose the contrary, that x $\notin \bigcap$ { W | W is pre-+-closed ans A $\subseteq$ W}. Then, x $\notin$ W for some pre-+-closed set W, so x $\in$ X $-$ W for some pre-+-open set X $-$ W. So, (X$-$ W) $\cap$ A = $\emptyset$ for some pre-+-open set X $-$ W containing x and this is a contradiction. Therefore, x $\notin$ { W | W is pre-+-closed ans A $\subseteq$ W}. Conversely, let y $\in$ x $\notin \bigcap${ W | W is pre-+-closed ans A $\subseteq$ W}. Then, y $\in$W for all pre-+-closed set W containing A. Now, suppose that y $\notin$ $\mathrm{Cl}_{p+(A)}$. Then, there exits pre-+-open set V containing y such that V $\cap$ A = $\emptyset$. Therefore, X $-$ V is pre-+-closed set containing A and y $\notin$ X $-$ V and this is a contradiction. Therefore, y $\in$ $\mathrm{Cl}_{p+(A)}$.

The proof of (3) and (4) are followed directly from the Definition 3.13. (5) and (6) are followed by applying part (4) of this Theorem. □

Theorem 3.15. *Let (X, τ) be a topological space and A, B $\subseteq$ X. Then, the following statements hold:*

(1) *If $A \subseteq B$, then $Int_{p+(A)} \subseteq Int_{p+(B)}$.*

(2) $Int_{p+(A)} \cup Int_{p+(A)} \subseteq Int_{p+(A\cup B)}$.

(3) $Int_{p+(A\cap B)} \subseteq Int_{p+(A)} \cap Int_{p+(B)}$.

Proof. The proof of (1) is followed directly from the Definition 3.11. (2) and (3) are followed by applying part (1) of this Theorem. □

Theorem 3.16. *Let (X, τ) be a topological space and A, B $\subseteq$ X. Then, the following statements hold:*

(1) $Int_{p+(X\setminus A)} = X \setminus Cl_{p+(A)}$

(2) $Cl_{p+(X\setminus A)} = X \setminus Int_{p+(A)}$

(3) $X \setminus Cl_{p+(X\setminus A)} = Int_{p+(A)}$

(4) $X \setminus Int_{p+(X\setminus A)} = Cl_{p+(A)}$

(5) $x \in Int_{p+(A)}$ *if and only if there exists a pre-+-open set M such that* $x \in M \subseteq A$.

Proof. Omitted. □

Theorem 3.17. *Let A be a subset of a topological space (X, τ). Then,* $x \in Cl_{p+(A)}$ *if and only if for every pre-+-open subset M of X containing x,* $A \cap M \neq \emptyset$.

Proof. Let x ∈ $\mathrm{Cl}_{p+(A)}$ and suppose that M ∩ A = ∅ for some *pre*-+-open set M which contains x. Then, (X \ M) is pre-+-closed and A ⊂ (X \ M), thus $\mathrm{Cl}_{p+(A)}$ ⊂ (X \ M). But this implies that x ∈ (X \ M), a contradiction. Thus, A ∩ M ≠ ∅.
Conversely, let A ⊆ X and x ∈ X such that for each *pre*-+-open set M_1 which contains x, $M_1 \cap \neq \emptyset$. If x ∉ $\mathrm{Cl}_{p+(A)}$, there is a *pre*-+-closed set F such that A ⊆ F and x ∉ F. Then, (X \ F) is a pre-+-open set with x ∈ (X \ F), and thus (X \ F) ∩ A ≠ ∅, which is a contradiction. □

Theorem 3.18. *Let (X, τ) be a topological space M ⊆ X. Then M is pre-+-open if and only if for each s ∈ M, there exists a pre-+-open set D such that s ∈ D ⊆ M.*

Proof. It is obvious. □

Definition 3.19. *A function f : (K, τ) → (L, σ) is said to be pre-+-irresolute if* $f^{-1}(A)$ *is pre-+-open set in (K, τ) for every pre-+-open set A of (L, σ).*

Theorem 3.20. *Let f : (K, τ) → (L, σ) be a function, then the following statements are equivalent:*

(1) *f is pre-+-irresolute.*

(2) *f($Cl_{p+(A)}$)* $\subseteq$ *$Cl_{p+(f(A))}$ holds for every subset A of K.*

(3) *f^{-1}(A) is pre-+-closed set in K, for every pre-+-closed subset A of L.*

Proof. (2) $\Rightarrow$ (3): Let A be a pre-+-closed set in L, then $\mathrm{Cl}_{p+(A)}$ = A. By using (2), we have f($\mathrm{Cl}_{p+f}{}^{-1}$(A))$\subseteq$ $\mathrm{Cl}_{p+(A)}$ = A. Thus, ($\mathrm{Cl}_{p+f}{}^{-1}$(A))$\subseteq$ f^{-1}(A) and hence f^{-1}(A) is pre-+-closed in K.

(3) $\Rightarrow$ (2): If A $\subseteq$ K, then $\mathrm{Cl}_{p+(f(A))}$ is pre-+-closed in L and by (3) f^{-1}($\mathrm{Cl}_{p+(f(A))}$) is pre-+-closed in K. Furthermore, A $\subseteq$ f^{-1}(f(A)) $\subseteq$ f^{-1}($\mathrm{Cl}_{p+(f(A))}$). Thus, $\mathrm{Cl}_{p+(A)}$ $\subseteq$ f^{-1}($\mathrm{Cl}_{p+(f(A))}$), consequently, f($\mathrm{Cl}_{p+(A)}$) $\subseteq$ f(f^{-1}($\mathrm{Cl}_{p+(f(A))}$)) $\subseteq$ $\mathrm{Cl}_{p+(f(A))}$.

(3) $\Leftrightarrow$ (1): Obvious. □

Definition 3.21. *A function f : X $\rightarrow$ Y is said to be pre-+-continuous at a point x $\in$ X if for each open subset K of Y containing f(x), there exists a pre-+-open subset L of X containing x such that f(L) $\subseteq$ K. The function f is said to be pre-+-continuous if it has this property at each x $\in$ X.*

Theorem 3.22. *A function f : X $\rightarrow$ Y is pre-+-continuous if and only if the inverse image of every open set in Y is pre-+-open in X.*

Proof. Let f be pre-+-continuous and K be any open set in Y. If f^{-1}(K) = $\emptyset$, then f^{-1}(K) is a pre-+-open set in X but if f^{-1}(K) $\neq$ $\emptyset$, then there exists x $\in$ f^{-1}(K) which implies f(x) $\in$ K. Since f is pre-+-continuous, then there exists a pre-+-open set L in X containing x such that f(L) $\subseteq$ K. This implies that x $\in$ L $\subseteq$ f^{-1}(K) and hence f^{-1}(K) is pre-+-open.

Conversely, let K be any open set in Y containing f(x), then x $\in$ f^{-1}(K) and by hypothesis f^{-1}(K) is a pre-+-open set in X containing x, so f(f^{-1}(K)) $\subseteq$ K. Thus, f is pre-+-continuous. □

Theorem 3.23. *For a function f : $X \to Y$, the following statements are equivalent:*

(1) *f is pre-+-continuous.*

(2) *$f^{-1}(K)$ is a pre-+-open set in X, for each open subset K of Y.*

(3) *$f^{-1}(F)$ is a pre-+-closed set in X, for each closed subset F of Y.*

(4) *$f(Cl_{p+(A)}) \subseteq Cl(f(A))$, for each subset A of X.*

(5) *$Cl_{p+(f}{}^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset B of Y.*

(6) *$f^{-1}(Int(B)) \subseteq Int_{p+(f}{}^{-1}(B))$, for each subset B of Y.*

Proof. (1) ⇒ (2): Directly from Theorem 3.22.

(2) ⇒ (3): Let F be any closed subset of Y. Then, $Y \setminus F$ is a open subset of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is a pre-+-open set in X and hence $f^{-1}(F)$ is a pre-+-closed set in X.

(3) ⇒ (4): Let A be any subset of X. Then, $f(A) \subseteq Cl(f(A))$ and $Cl(f(A))$ is a closed set in Y. Hence, $A \subseteq f^{-1}(Cl(f(A)))$. By (3), we have $f^{-1}(Cl(f(A)))$ is a pre-+-closed set in X. Therefore, $Cl_{p+(A)} \subseteq f^{-1}(Cl(f(A)))$. Hence, $f(Cl_{p+(A)}) \subseteq Cl(f(A))$.

(4) ⇒ (5): Let B be any subset of Y. Then, $f^{-1}(B)$ is a subset of X. By (4), we have $f(Cl_{p+(f}{}^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) \subseteq Cl(B)$. Hence, $Cl_{p+(f}{}^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(5) ⇔ (6): Let B be any subset of Y. Then, apply (5) to $Y \setminus B$ we obtain $Cl_{p+(f}{}^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B)) \Leftrightarrow Cl_{p+(X} \setminus f^{-1}(B)) \subseteq f^{-1}(X \setminus Int(B)) \Leftrightarrow X \setminus Int_{p+(f}{}^{-1}(B)) \subseteq X \setminus f^{-1}(int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq Int_{p+(f}{}^{-1}(B))$. Thus, $f^{-1}(Int(B))) \subseteq Int_{p+(f}{}^{-1}(B))$.

(6) ⇒ (1): Let $x \in X$ and K be any open subset of Y containing f(x). By (6), we have $f^{-1}(Int(K)) \subseteq Int_{p+(f}{}^{-1}(K))$ implies that $f^{-1}(K) \subseteq Int_{p+(f}{}^{-1}(K))$. Hence, $f^{-1}(K)$ is a pre-+-open set in X which contains x and clearly $f(f^{-1}(K)) \subseteq K$. Thus, f is pre-+-continuous. □

Definition 3.24. *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be a generalized pre-+-closed set or simply gp-+-closed set if $Cl_{p+(A)} \subseteq U$, whenever $A \subseteq U$ and U is a open set.*
The complement of a gp-+-closed set is called gp-+-open set.
The collection of all gp-+-closed sets and gp-+-open sets are denoted by gp+ (X, τ) and gp+ (X, τ), respectively.

Proposition 3.25. *Every pre-+-closed set is gp-+-closed set.*

Proof. The proof is followed by the Definition 3.24. □

The following example shows that the converse of the above Proposition, it is not always true.

Example 3.26. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, c\}\}$. Then pre-+-closed sets are $\emptyset$, X, $\{a\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$; gp-+-closed sets are power set of X. Here $\{b\}$ is a gp-+-closed set but it is not a pre-+-closed set.*

Theorem 3.27. *Let A be a gp-+-closed subset of X. Then, $Cl_{p+(A)} - A$ does not contain any non-empty closed sets.*

Proof. Let F be a closed set of X such that $\mathrm{F} \subseteq \mathrm{Cl}_{p+(A)} - \mathrm{A}$. Since $\mathrm{X} - \mathrm{F}$ is a open set, then $\mathrm{A} \subseteq \mathrm{X} - \mathrm{F}$ and A is gp+-closed, it follows $\mathrm{Cl}_{p+(A)} \subseteq \mathrm{X} - \mathrm{F}$, in consequence $\mathrm{F} \subseteq \mathrm{X} - \mathrm{Cl}_{p+(A)}$. This implies that $\mathrm{F} \subseteq (\mathrm{X} - \mathrm{Cl}_{p+(A)}) \cap (\mathrm{Cl}_{p+(A)} - \mathrm{A}) = \emptyset$, therefore $\mathrm{F} = \emptyset$. □

Corollary 3.28. *Let A be a gp+-closed set. Then, A is pre-+-closed if and only if $Cl_{p+(A)} - A$ is a closed set.*

Proof. Let A be gp-+-closed set. If A is pre-+-closed, it has $\mathrm{Cl}_{p+(A)} - \mathrm{A} = \emptyset$ which is a closed set. Conversely, let $\mathrm{Cl}_{p+(A)} - \mathrm{A}$ be a closed set. Then, by the Theorem 3.27, $\mathrm{Cl}_{p+(A)} - \mathrm{A}$ does not contain any non-empty closed set and $\mathrm{Cl}_{p+(A)}$ is a closed set of itself. Thus, $\mathrm{Cl}_{p+(A)} - \mathrm{A} = \emptyset$. Therefore, $\mathrm{A} = \mathrm{Cl}_{p+(A)}$, in consequence A is a pre-+-closed set. □

Corollary 3.29. *Let A be an open set and gp-+-closed set. Then, A ∩ J is gp-+-closed set whenever pre-+-closed set J of X.*

Proof. Since A is gp-+-closed and open set, then $\mathrm{Cl}_{p+(A)} \subseteq \mathrm{A}$ and so A is a pre-+-closed. Therefore, A ∩ J is pre-+-closed set of X and this implies that A ∩ J is gp-+-closed set of X. □

Theorem 3.30. *Let (X, τ) be a topological space and A, B ⊆ X. If A is a gp-+-closed set and B is any set such that $A \subseteq B \subseteq Cl_{p+(A)}$, then B is a gp-+-closed set of X.*

Proof. Let B ⊆ V where V is an open set of X. Since A is a gp-+-closed set and A ⊆ V , then $\mathrm{Cl}_{p+(A)} \subseteq \mathrm{V}$ and so $\mathrm{Cl}_{p+(A)} = \mathrm{Cl}_{p+(B)}$. Therefore, $\mathrm{Cl}_{p+(B)} \subseteq \mathrm{V}$ and hence B is a gp-+-closed set of X. □

Theorem 3.31. *Let (X, τ) be a topological space and A ⊂ X. A is a gp-+-open set if and only if $J \subseteq Int_{p+(A)}$ whenever J closed set and J ⊆ A.*

Proof. Let A be a gp-+-open set and let J ⊆ A where J is a closed set. Then, X − A is a gp-+-closed set contained in the open set X − J. Therefore, $\mathrm{Cl}_{p+}(\mathrm{X}-\mathrm{A}) \subseteq \mathrm{X} - \mathrm{J}$ and $\mathrm{X} - \mathrm{Int}_{p+(A)} \subseteq \mathrm{X} - \mathrm{J}$. In consequence, $\mathrm{J} \subseteq \mathrm{Int}_{p+(A)}$.
Conversely, if A is a closed set with $\mathrm{J} \subseteq \mathrm{Int}_{p+(A)}$ and J ⊆ A, then $\mathrm{X} - \mathrm{Int}_{p+(A)} \subseteq \mathrm{X} - \mathrm{J}$. Therefore, $\mathrm{Cl}_{p+}(\mathrm{X}-\mathrm{A}) \subseteq \mathrm{X} - \mathrm{J}$. Hence, X − A is a gp-+-closed set and A is a gp-+-open set of X. □

Theorem 3.32. *Every α-+-continuous is pre-+-continous but not conversely.*

Proof. The proof follows from Proposition 3.9. □

Definition 3.33. *A function $f : (K, \tau) \to (L, \sigma)$ is said to be gp-+-continuous if $f^{-1}(A)$ is gp-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 3.34. *Every pre-+-continuous is gp-+-continuous but not conversely.*

Proof. The proof follows from Proposition 3.25. □

Definition 3.35. *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be a regular generalized pre-+-closed set or simply $rg\alpha$-+-closed set if $Cl_{p+(A)} \subseteq U$, whenever $A \subseteq U$ and U is a regular open set.*
The complement of a rgp-+-closed set is called rgp-+-open set.
The collection of all rgp-+-closed sets and rgp-+-open sets are denoted by rgp+(X, τ) and rgp+(X, τ),respectively.

Proposition 3.36. *Every closed set is rgp-+-closed set.*

Proof. Let B be any closed set of X such that B $\subseteq$ V . where V is a regular open set. Since $\mathrm{Cl}_{p+(B)} \subseteq \mathrm{Cl}(\mathrm{B}) = \mathrm{B}$. Therefore, $\mathrm{Cl}_{p+(B)} \subseteq$ V . In consequence, B is a rgp-+-closed set. □
The following example shows that the converse of the above Theorem need not be true.

Example 3.37. *Let X and $\tau(X)$ as in the Example 3.8. Then rgp-+-closed sets are power set of X. Here, $\{a, b\}$ is rgp-+-closed set but it is not closed set.*

Proposition 3.38. *Every pre-+-closed set is rgp-+-closed set.*

Proof. Let B be any pre-+-closed set of X such that V is any regular open set containing B. Since B is a pre-+- closed set, then $\mathrm{Cl}_{p+(B)} = \mathrm{B}$. Therefore, $\mathrm{Cl}_{p+(B)} \subseteq V$. Hence, B is a rgp-+-closed set. □

The following example shows that the converse of the above Theorem need not be true.

Example 3.39. *Let X and $\tau(X)$ as in the Example 3.37. Here, $\{a, b, c\}$ is rgp-+-closed set but it is not pre-+-closed set.*

Proposition 3.40. *Every gp-+-closed set is rgp-+-closed set but not conveserly.*

Proof. If A is a gp-+-closed subset of (X, τ) and G is any regular open set containing A, since every regular open set is open, we have $G \supseteq \mathrm{Cl}_{p+(A)}$. Hence A is rgp-+-closed in (X, τ). □

Example 3.41. *Let X and $\tau(X)$ as in the Example 2.38. Then gp-+-closed sets are $\emptyset$, X, $\{a\}$, $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d$, $\{b, c, d\}$; rgp-+-closed sets are power of X. Here $\{a, b\}$ is rgp-+-closed set but it is not a gp-+-closed set.*

Definition 3.42. *A function $f : (K, \tau) \to (L, \sigma)$ is said to be regular generalized pre-+-continuous (berifly, rgp-+-continuous)if $f^{-1}(A)$ is rgp-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 3.43. *Every continuous is rgp-+-continuous but not conversely.*

Proof. The proof follows from Proposition 3.36. □

Theorem 3.44. *Every pre-+-continuous is rgp-+-continuous but not conversely.*

Proof. The proof follows from Proposition 3.38. □

Proposition 3.45. *Every gp-+-continuous is rgp-+-continuous but not conveserly.*

Proof. The proof follows from Proposition 3.40. □

4. β-+-open sets and generalized β-+-closed sets

Definition 4.1. *Let A subset A of a space (X, τ) is said to be β-+-open if $A^+ \subseteq$ $Cl(Int(Cl(A^+)))$.*

The complement of β-+-open sets is called β-+-closed sets.
The collection of all β-+-open sets and β-+-closed sets are denoted by $\beta+O(X, \tau)$ and $\beta+C(X, \tau)$ respectively.

Theorem 4.2. *Every closed set is β-+-open.*

Proof. Let A be a closed set of (X, τ). Since A is a closed set, then Cl(A) = A and so $A^+ = \mathrm{Cl(A)} - \mathrm{A} = \mathrm{A} \cap A^c = \phi$. Now, $\phi \subseteq \mathrm{Cl(Int(Cl(A^+)))} = Cl(Int(Cl(\phi))) = \phi$. This shows that A is β-+-open. □

Remark 4.3. *The following example shows that open sets is independent of β-+-open sets.*

Example 4.4. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$. Then β-+-open sets are $\emptyset$, X, $\{b\}$, $\{b, c\}$. Here $\{a, c\}$ is a open set but it is not a β-+-open set. Also it is clear that $\{b, c\}$ is a β-+-open set but it is not a open set.*

Remark 4.5. *The following example shows that the notion of β-open set and β-+-open set are independent.*

Example 4.6. *Let X and $\tau(X)$ in the Example 4.4. Then β-open sets are $\emptyset$, X, $\{a\}$, $\{a,\ b\}$, $\{a,\ c\}$. Here $\{a,\ b\}$ is a β-+-open set but it is not a β-+-open set. Also it is clear that $\{b,\ c\}$ is a β-+-open set but it is not a β-open set.*

Remark 4.7. *The union(intersection) of any two β-+-open sets in not β-+-open. For,*

Example 4.8. *Let $X = \{a,\ b,\ c,\ d\}$ and $\tau = \{\emptyset,\ X,\ \{a\},\ \{b,\ c\},\ \{a,\ b,\ c\}\}$. Then β-+-open sets are $\emptyset$, X, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,\ b\}$, $\{a,\ c\}$, $\{a,\ d\}$, $\{b,\ d\}$, $\{c,\ d\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d\}$, $\{b,\ c,\ d\}$. (i) Let $A = \{b\}$ and $B = \{c\}$ are β-+-open sets, but $A \cup B = \{b,\ c\}$ is not β-+-open set. (ii) Let $A = \{a,\ b\}$ and $B = \{a,\ d\}$ are β-+-open sets, but $A \cap B = \{a\}$ is not β-+-open set.*

Proposition 4.9. *Every b-+-open is β-+-open*

Proof. Let A be b-+-open, then A $\subseteq$ Int(Cl(A$^+$))$\cup$ Cl(Int(A$^+$))$\subseteq$ Cl(Int(Cl(A$^+$)))$\cup$ Cl(Int(A$^+$))

$\subseteq$ Cl(Int(Cl(A$^+$)))$\cup$ Int(A$^+$)$\subseteq$ Cl(Int(Cl(A$^+$)))$\cup$ Int(Cl(A$^+$))

$\subseteq$ Cl(Int(Cl(A$^+$))).

Thus, A is β-+-open. □

The converse of the above Theorem need not be true in general as it shown below.

Example 4.10. *Let X and $\tau(X)$ in the Example 4.8. b-+-open sets are $\emptyset$, X, $\{d\}$, $\{a,\ d\}$, $\{b,\ d\}$, $\{c,\ d\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d\}$, $\{b,\ c,\ d\}$. Here $\{a,\ b\}$ is β-+-open set but it is not b-+-open set.*

Definition 4.11. *Let $(X,\ \tau)$ be a topological space and $A \subset X$. An element $x \in A$ is said to be β-+-interior point of A if there exits a β-+-open set U such that $x \in U \subseteq A$. The set of all β-+-interior points of A is said to be β+-interior of A an it is denoted by $Int_{\beta+(A)}$.*

Theorem 4.12. *Let (X, τ) be a topological space and $A \subset X$. Then, A is β-+-open if and only if $A = Int_{\beta+(A)}$.*

Proof. Let A be a β-+-open set. Then, $A \subseteq A$ and this implies that $A \in \{U \mid U$ is β-+-open and $U \subset A\}$. Since union of this collection is in A. Therefore, $A = \text{Int}_{\beta+(A)}$. Conversely, suppose that $A = \text{Int}_{\beta+(A)}$. Hence, A is β-+-open. □

Definition 4.13. *Let $A \subset X$. Then $x \in X$ is β-+-adherent to A if $U \cap A \notin \emptyset$ for every β-+-open set U containing x. The set of all β-+-adherent points of A is said to be β-+-closure of A and it is denoted by $Cl_{\beta+(A)}$.*

Theorem 4.14. *Let (X, τ) be a topological space and A, $B \subseteq X$. Then, the following statements hold:*

(1) *$A \subseteq Cl_{\beta+(A)}$.*
(2) *$Cl_{\beta+(A)}$ is the smallest β-+-closed set containing A, that is $Cl_{\beta+(A)} = \bigcap \{ W \mid W$ is β-+-closed ans $A \subseteq W\}$.*
(3) *A is β-+-closed if and only if $A = Cl_{\beta+(A)}$.*
(4) *If $A \subseteq B$, then $Cl_{\beta+(A)} \subseteq Cl_{\beta+(B)}$.*
(5) *$Cl_{\beta+(A)} \cup Cl_{\beta+(A)} \subseteq Cl_{\beta+(A\cup B)}$.*
(6) *$Cl_{\beta+(A\cap B)} \subseteq Cl_{\beta+(A)} \cap Cl_{\beta+(B)}$.*

Proof. (1) Let $x \in A$ and suppose that $x \notin \text{Cl}_{\beta+(A)}$. Then, there exits β-+-open set V containing x such that $V \cap A = \emptyset$ and this is a contradiction. Therefore, $x \in \text{Cl}_{\beta+(A)}$.
(2) Let $x \in \text{Cl}_{\beta+(A)}$. Then, $V \cap A \neq \emptyset$ for every β-+-open set V containing x. Now, suppose the contrary, that $x \notin \bigcap \{ W \mid W$ is β-+-closed ans $A \subseteq W\}$. Then, $x \notin W$ for some β-+-closed set W, so $x \in X - W$ for some β-+-open set $X - W$. So, $(X - W) \cap A = \emptyset$ for some β-+-open set $X - W$ containing x and this is a contradiction.

Therefore, x $\notin$ { W | W is β-+-closed ans A $\subseteq$ W}. Conversely, let y $\in$ x $\notin \bigcap${ W | W is β-+-closed ans A $\subseteq$ W}. Then, y $\in$W for all β-+-closed set W containing A. Now, suppose that y $\notin$ $\mathrm{Cl}_{\beta+(A)}$. Then, there exits β-+-open set V containing y such that V $\cap$ A $= \emptyset$. Therefore, X $-$ V is β-+-closed set containing A and y $\notin$ X $-$ V and this is a contradiction. Therefore, y $\in$ $\mathrm{Cl}_{\beta+(A)}$.

The proof of (3) and (4) are followed directly from the Definition 4.13. (5) and (6) are followed by applying part (4) of this Theorem. □

Theorem 4.15. *Let (X, τ) be a topological space and A, B $\subseteq$ X. Then, the following statements hold:*

(1) *If $A \subseteq B$, then $Int_{\beta+(A)} \subseteq Int_{\beta+(B)}$.*

(2) $Int_{\beta+(A)} \cup Int_{\beta+(A)} \subseteq Int_{\beta+(A\cup B)}$.

(3) $Int_{\beta+(A\cap B)} \subseteq Int_{\beta+(A)} \cap Int_{\beta+(B)}$.

Proof. The proof of (1) is followed directly from the Definition 4.11. (2) and (3) are followed by applying part (1) of this Theorem. □

Theorem 4.16. *Let (X, τ) be a topological space and A, B $\subseteq$ X. Then, the following statements hold:*

(1) $Int_{\beta+(X\setminus A)} = X \setminus Cl_{\beta+(A)}$.

(2) $Cl_{\beta+(X\setminus A)} = X \setminus Int_{\beta+(A)}$.

(3) $X \setminus Cl_{\beta+(X\setminus A)} = Int_{\beta+(A)}$.

(4) $X \setminus Int_{\beta+(X\setminus A)} = Cl_{\beta+(A)}$.

(5) *$x \in Int_{\beta+(A)}$ if and only if there exists a β-+-open set M such that $x \in M \subseteq A$.*

Proof. . □

Theorem 4.17. *Let A be a subset of a topological space (X, τ). Then,* $x \in Cl_{\beta+(A)}$ *if and only if for every β-+-open subset M of X containing x,* $A \cap M \neq \emptyset$*.*

Proof. Let $x \in Cl_{\beta+(A)}$ and suppose that $M \cap A = \emptyset$ for some β-+-open set M which contains x. Then, $(X \setminus M)$ is β-+-closed and $A \subset (X \setminus M)$, thus $Cl_{\beta+(A)} \subset (X \setminus M)$. But this implies that $x \in (X \setminus M)$, a contradiction. Thus, $A \cap M \neq \emptyset$.
Conversely, let $A \subseteq X$ and $x \in X$ such that for each β-+-open set M_1 which contains x, $M_1 \cap \neq \emptyset$. If $x \notin Cl_{\beta+(A)}$, there is a β-+-closed set F such that $A \subseteq F$ and $x \notin F$. Then, $(X \setminus F)$ is a β-+-open set with $x \in (X \setminus F)$, and thus $(X \setminus F) \cap A \neq \emptyset$, which is a contradiction. □

Theorem 4.18. *Let (X, τ) be a topological space* $M \subseteq X$*. Then M is β-+-open if and only if for each* $s \in M$*, there exists a β-+-open set D such that* $s \in D \subseteq M$*.*

Proof. It is obvious. □

Definition 4.19. *A function f : (K, τ) → (L, σ) is said to be β-+-irresolute if* $f^{-1}(A)$ *is β-+-open set in (K, τ) for every β-+-open set A of (L, σ).*

Theorem 4.20. *Let f : (K, τ) → (L, σ) be a function, then the following statements are equivalent:*

(1) *f is β-+-irresolute.*
(2) $f(Cl_{\beta+(A)}) \subseteq Cl_{\beta+(f(A))}$ *holds for every subset A of K.*
(3) $f^{-1}(A)$ *is β-+-closed set in K, for every β-+-closed subset A of L.*

Proof. (2) ⇒ (3): Let A be a β-+-closed set in L, then $Cl_{\beta+(A)} = A$. By using (2), we have $f(Cl_{\beta+f}{}^{-1}(A)) \subseteq Cl_{\beta+(A)} = A$. Thus, $(Cl_{\beta+f}{}^{-1}(A)) \subseteq f^{-1}(A)$ and hence $f^{-1}(A)$ is β-+-closed in K.

(3) $\Rightarrow$ (2): If A $\subseteq$ K, then $\text{Cl}_{\beta+(f(A))}$ is β-+-closed in L and by (3) $\text{f}^{-1}(\text{Cl}_{\beta+(f(A))})$ is β-+-closed in K. Furthermore, A $\subseteq \text{f}^{-1}(\text{f(A)}) \subseteq \text{f}^{-1}(\text{Cl}_{\beta+(f(A))})$. Thus, $\text{Cl}_{\beta+(A)} \subseteq \text{f}^{-1}(\text{Cl}_{\beta+(f(A))})$, consequently, $\text{f}(\text{Cl}_{\beta+(A)}) \subseteq \text{f}(\text{f}^{-1}(\text{Cl}_{\beta+(f(A))})) \subseteq \text{Cl}_{\beta+(f(A))}$.

(3) $\Leftrightarrow$ (1): Obvious. □

Definition 4.21. *A function $f : X \rightarrow Y$ is said to be β-+-continuous at a point $x \in X$ if for each open subset K of Y containing f(x), there exists a β-+-open subset L of X containing x such that f(L) $\subseteq$ K. The function f is said to be β-+-continuous if it has this property at each $x \in X$.*

Theorem 4.22. *A function $f : X \rightarrow Y$ is β-+-continuous if and only if the inverse image of every open set in Y is β-+-open in X.*

Proof. Let f be β-+-continuous and K be any open set in Y. If $\text{f}^{-1}(\text{K}) = \emptyset$, then $\text{f}^{-1}(\text{K})$ is a β-+-open set in X but if $\text{f}^{-1}(\text{K}) \neq \emptyset$, then there exists $\text{x} \in \text{f}^{-1}(\text{K})$ which implies f(x) $\in$ K. Since f is β-+-continuous, then there exists a β-+-open set L in X containing x such that f(L) $\subseteq$ K. This implies that $\text{x} \in \text{L} \subseteq \text{f}^{-1}(\text{K})$ and hence $\text{f}^{-1}(\text{K})$ is β-+-open.

Conversely, let K be any open set in Y containing f(x), then $\text{x} \in \text{f}^{-1}(\text{K})$ and by hypothesis $\text{f}^{-1}(\text{K})$ is a β-+-open set in X containing x, so $\text{f}(\text{f}^{-1}(\text{K})) \subseteq$ K. Thus, f is β-+-continuous. □

Theorem 4.23. *For a function $f : X \rightarrow Y$, the following statements are equivalent:*

(1) *f is β-+-continuous.*

(2) *f^{-1}(K) is a β-+-open set in X, for each open subset K of Y.*

(3) *f^{-1}(F) is a β-+-closed set in X, for each closed subset F of Y.*

(4) *$f(Cl_{\beta+(A)}) \subseteq Cl(f(A))$, for each subset A of X.*

(5) *$Cl_{\beta+(f}{}^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset B of Y.*

(6) *f^{-1}(Int(B)) $\subseteq$ $Int_{\beta+(f}{}^{-1}$(B)), for each subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Directly from Theorem 4.22.

(2) $\Rightarrow$ (3): Let F be any closed subset of Y. Then, Y $\setminus$ F is a open subset of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is a β-+-open set in X and hence $f^{-1}(F)$ is a β-+-closed set in X.

(3) $\Rightarrow$ (4): Let A be any subset of X. Then, f(A) $\subseteq$ Cl(f(A)) and Cl(f(A)) is a closed set in Y. Hence, A $\subseteq$ f^{-1}(Cl(f(A))). By (3), we have f^{-1}(Cl(f(A))) is a β-+-closed set in X. Therefore, $Cl_{\beta+(A)} \subseteq f^{-1}$(Cl(f(A))). Hence, f($Cl_{\beta+(A)}$) $\subseteq$ Cl(f(A)).

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, f^{-1}(B) is a subset of X. By (4), we have f($Cl_{\beta+(f}{}^{-1}$(B)))$\subseteq$ Cl(f(f^{-1}(B))) $\subseteq$ Cl(B). Hence, $Cl_{\beta+(f}{}^{-1}$(B))$\subseteq$ f^{-1}(Cl(B)).

(5) $\Leftrightarrow$ (6): Let B be any subset of Y. Then, apply (5) to Y $\setminus$ B we obtain $Cl_{\beta+(f}{}^{-1}$(Y $\setminus$ B))$\subseteq$ f^{-1}(Cl(Y $\setminus$ B)) $\Leftrightarrow$ $Cl_{\beta+(X}\setminus$ f^{-1}(B))$\subseteq$ f^{-1}(X $\setminus$ Int(B)) $\Leftrightarrow$ X $\setminus$ $Int_{\beta+(f}{}^{-1}$(B))$\subseteq$ X $\setminus$ f^{-1}(int(B)) $\Leftrightarrow$ f^{-1}(Int(B)) $\subseteq$ $Int_{\beta+(f}{}^{-1}$(B)). Thus, f^{-1}(Int(B))) $\subseteq$ $Int_{\beta+(f}{}^{-1}$(B)).

(6) $\Rightarrow$ (1): Let x $\in$ X and K be any open subset of Y containing f(x). By (6), we have f^{-1}(Int(K)) $\subseteq$ $Int_{\beta+(f}{}^{-1}$(K)) implies that f^{-1}(K) $\subseteq$ $Int_{\beta+(f}{}^{-1}$(K)). Hence, f^{-1}(K) is a β-+-open set in X which contains x and clearly f(f^{-1}(K)) $\subseteq$ K. Thus, f is β-+-continuous. □

Definition 4.24. *Let (X, τ) be a topological space and A $\subseteq$ X. Then, A is said to be a generalized β-+-closed set or simply gβ-+-closed set if $Cl_{\beta+(A)}$ $\subseteq$ U, whenever A $\subseteq$ U and U is a open set.*

The complement of a gβ-+-closed set is called gβ-+-open set.

The collection of all gβ-+-closed sets and gβ-+-open sets are denoted by gβ+ (X, τ) and gβ+ (X, τ), respectively.

Proposition 4.25. *Every β-+-closed set is gβ-+-closed set.*

Proof. The proof is followed by the Definition 4.24. □

The following example shows that the converse of the above Proposition, it is not always true.

Example 4.26. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then β-+-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$; $g\beta$-+-closed sets are power of set X. Here $\{b, c\}$ is a $g\beta$-+-closed set but it is not a β-+-closed set.*

Theorem 4.27. *Let A be a $g\beta$-+-closed subset of X. Then, $Cl_{\beta+(A)} - A$ does not contain any non-empty closed sets.*

Proof. Let F be a closed set of X such that $F \subseteq Cl_{\beta+(A)} - A$. Since $X - F$ is a open set, then $A \subseteq X - F$ and A is $g\beta$-+-closed, it follows $Cl_{\beta+(A)} \subseteq X - F$, in consequence $F \subseteq X - Cl_{\beta+(A)}$. This implies that $F \subseteq (X - Cl_{\beta+(A)}) \cap (Cl_{\beta+(A)} - A) = \emptyset$, therefore $F = \emptyset$. □

Corollary 4.28. *Let A be a $g\beta$-+-closed set. Then, A is β-+-closed if and only if $Cl_{\beta+(A)} - A$ is a closed set.*

Proof. Let A be $g\beta$-+-closed set. If A is β-+-closed, it has $Cl_{\beta+(A)} - A = \emptyset$ which is a closed set. Conversely, let $Cl_{\beta+(A)} - A$ be a closed set. Then, by the Theorem 4.27, $Cl_{\beta+(A)} - A$ does not contain any non-empty closed set and $Cl_{\beta+(A)}$ is a closed set of itself. Thus, $Cl_{\beta+(A)} - A = \emptyset$. Therefore, $A = Cl_{\beta+(A)}$, in consequence A is a β-+-closed set. □

Corollary 4.29. *Let A be an open set and $g\beta$-+-closed set. Then, $A \cap J$ is $g\beta$-+-closed set whenever β-+-closed set J of X.*

Proof. Since A is $g\beta$-+-closed and open set, then $\mathrm{Cl}_{\beta+(A)} \subseteq$ A and so A is a β-+-closed. Therefore, A $\cap$ J is β-+-closed set of X and this implies that A $\cap$ J is $g\beta$-+-closed set of X. □

Theorem 4.30. *Let (X, τ) be a topological space and A, B $\subseteq$ X. If A is a $g\beta$-+-closed set and B is any set such that A $\subseteq$ B $\subseteq$ $Cl_{\beta+(A)}$, then B is a $g\beta$-+-closed set of X.*

Proof. Let B $\subseteq$ V where V is an open set of X. Since A is a $g\beta$-+-closed set and A $\subseteq$ V , then $\mathrm{Cl}_{\beta+(A)} \subseteq$ V and so $\mathrm{Cl}_{\beta+(A)} = \mathrm{Cl}_{\beta+(B)}$. Therefore, $\mathrm{Cl}_{\beta+(B)} \subseteq$ V and hence B is a $g\beta$-+-closed set of X. □

Theorem 4.31. *Let (X, τ) be a topological space and A $\subset$ X. A is a $g\beta$-+-open set if and only if J $\subseteq$ $Int_{\beta+(A)}$ whenever J closed set and J $\subseteq$ A.*

Proof. Let A be a $g\beta$-+-open set and let J $\subseteq$ A where J is a closed set. Then, X $-$ A is a $g\beta$-+-closed set contained in the open set X $-$ J. Therefore, $\mathrm{Cl}_{\beta+}$(X$-$A) $\subseteq$ X $-$ J and X $-$ $\mathrm{Int}_{\beta+(A)} \subseteq$ X $-$ J. In consequence, J $\subseteq \mathrm{Int}_{\beta+(A)}$.
Conversely, if A is a closed set with J $\subseteq \mathrm{Int}_{\beta+(A)}$ and J $\subseteq$ A, then X $-$ $\mathrm{Int}_{\beta+(A)} \subseteq$ X $-$ J. Therefore, $\mathrm{Cl}_{\beta+}$(X$-$A) $\subseteq$ X $-$ J. Hence, X $-$ A is a $g\beta$-+-closed set and A is a $g\beta$-+-open set of X. □

Proposition 4.32. *Every b-+-continous is β-+-continuous but not conversely.*

Proof. The proof follows from Proposition 4.9. □

Definition 4.33. *A function f : (K, τ) $\rightarrow$ (L, σ) is said to be $g\beta$-+-continuous if f^{-1}(A) is $g\beta$-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 4.34. *Every β-+-continuous is $g\beta$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 4.25. □

Definition 4.35. *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be a regular generalized β-+-closed set or simply rgβ-+-closed set if $Cl_{\beta+(A)} \subseteq U$, whenever $A \subseteq U$ and U is a regular open set.*
The complement of a rgβ+-closed set is called rgβ-+-open set.
The collection of all rgβ-+-closed sets and rgβ-+-open sets are denoted by rgβ+(X, τ) and rgβ+(X, τ),respectively.

Proposition 4.36. *Every closed set is rgβ-+-closed set.*

Proof. Let B be any closed set of X such that $B \subseteq V$. where V is a regular open set. Since $Cl_{\beta+(B)} \subseteq Cl(B) = B$. Therefore, $Cl_{\beta+(B)} \subseteq V$. In consequence, B is a rgβ-+-closed set. □
The following example shows that the converse of the above Theorem need not be true.

Example 4.37. *Let X and τ as in the example 4.8. Then rgβ-+-closed sets are power set of X. Here, $\{a, b, c\}$ is rgβ-+-closed set but it is not closed set.*

Proposition 4.38. *Every β-+-closed set is rgβ-+-closed set.*

Proof. Let B be any β-+-closed set of X such that V is any regular open set containing B. Since B is a β-+- closed set, then $Cl_{\beta+(B)} = B$. Therefore, $Cl_{\beta+(B)} \subseteq V$. Hence, B is a rgβ+-closed set. □
The following example shows that the converse of the above Theorem need not be true.

Example 4.39. *Let X and τ as in the example 4.37. Here $\{a, d\}$ is rgβ-+-closed set but it is not β-+-closed set.*

Proposition 4.40. *Every $g\beta$-+-closed set is $rg\beta$-+-closed set but not conveserly.*

Proof. If A is a $g\beta$-+-closed subset of (X, τ) and G is any regular open set containing A, since every regular open set is open, we have G $\supseteq$ $\mathrm{Cl}_{\beta+(A)}$. Hence A is $rg\beta$-+-closed in (X, τ). □

Example 4.41. *Let X and τ as in the example 2.38. Then $g\beta$-+-closed sets are $\emptyset$, X, $\{a\}$, $\{d\}$, $\{a,\ d\}$, $\{b,\ d\}$, $\{c,\ d\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d,\ \{b,\ c,\ d\}$; $rg\beta$-+-closed sets are power of X. Here $\{b\}$ is $rg\beta$-+-closed set but it is not a $g\beta$-+-closed set.*

Definition 4.42. *A function f : (K, τ) $\rightarrow$ (L, σ) is said to be regular generalized β-+-continuous (berifly, $rg\beta$-+-continuous) if $f^{-1}(A)$ is $rg\beta$-+-closed set in (K, τ) for every closed set A of (L, σ).*

Theorem 4.43. *Every continuous is $rg\beta$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 4.36. □

Theorem 4.44. *Every β-+-continuous is $rg\beta$-+-continuous but not conversely.*

Proof. The proof follows from Proposition 4.38. □

Proposition 4.45. *Every $g\beta$-+-continuous is $rg\beta$-+-continuous but not conveserly.*

Proof. The proof follows from Proposition 4.40. □

Conclusion

Every year different type of topological spaces are introduced by many topologist. Now a days available topologies are ideal topology, grill topology, bitopology, fuzzy topology, nano topology, nano ideal topology, micro topology, micro ideal topology, grill ideal topology and so on. In this paper we introduced a new operator of open sets and generalized closed sets in topological spaces. Also we characterize the relations between them and the related properties. In future, we have extended this work in various topological fields with some applications.

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