A new Approach for Maker-Breaker Domination Game in Graphs

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Abstract

The Maker –Breaker domination game in graphs is introduced as a natural counterpart to the Maker-Breaker domination game.The Maker-Breaker domination game is played on a graph G by Dominator and Staller. The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. In this paper we introduce the Maker-Breaker domination number $\gamma MB(G)$ of G as the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. If Staller plays first, then the corresponding invariant is denoted $\gamma' MB(G)$. Comparing the two invariants it turns out that they behave much diff erently than the related game domination numbers. The invariant $\gamma MB(G)$ is also compared with the domination number. Using the Erd˝os-Selfridge Criterion a large class of graphs G is found for which $\gamma MB(G) > \gamma(G)$ holds. Residual graphs are introduced and used to bound/determine $\gamma MB(G)$ and $\gamma' MB(G)$. Using residual graphs, $\gamma MB(T)$ and $\gamma' MB(T)$ are determined for an arbitrary tree. The invariants are also obtained for cycles and bounded for union of graphs. A list of problems for further investigations is given.

Key words: Maker-Breaker domination game,domination number,perfect matching,tree, cycle, union of graphs.

Introduction

The Maker-Breaker domination game (MBD game for short) was studied for the first time in [11]. The game is played on a graph G by two players. To be consistent with the naming from the usual and well-investigated domination game [3] .The players are named Dominator and Staller. They are selecting vertices alternatively, always selecting a vertex that has not yet been chosen. Dominator wins the MBD game on G if at some point the set of vertices already selected by him forms a dominating set of G, that is, a set D such that every vertex not in D has a neighbour in D. Otherwise Staller wins, that is, she wins if she is able to select all the vertices from the closed neighbourhood of some vertex. Just as the total domination game [15] (see also [2, 6, 16, 17]) followed the domination game, we introduce here the Maker-Breaker total domination game (MBD game for short) the Maker-Breaker game introduced in 1973 by Erd″os and Selfridge [12]. The game is played on a hypergraph H. One of the player, Maker, wins if he is able to select all the vertices of one of the hyperedges of H, while the other player, Breaker, wins if she is able to keep Maker from doing so. There is an abundant literature on this topic, see the books of Beck [1] and of Hefetz et al. [14] for related surveysand Maker-Breaker games are played on hypergraphs by two players called Maker and Breaker. They take turns and at each turn the current player selects a new vertex. Maker wins if at some point of the game he has selected all vertices from one of the hyperedges, while Breaker wins if she can keep him from doing it. See [1] and [16] for general introductions on this field.

Very recently, the Maker-Breaker domination game was introduced in [12]. The game is played on a graph G with two players named Dominator and Staller. These names were selected to emphasize the domination nature of the game and to be consistent with the usual domination game where these two names are standard by now. The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point, the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. Note that the Maker-Breaker domination game is a MakerBreaker game. Indeed, if for a graph G we build a hypergraph F with the same set of vertices as G, and in which the hyperedges are the dominating sets of G, then Dominator wins the Maker-Breaker domination game on G if and only if Maker wins the Maker-Breaker game on F. In several papers on Maker-Breaker games the authors were interested in the smallest number of moves needed for Maker to win, see [7, 8, 15]. Also, in [12] it was emphasized that when dealing with the Maker-Breaker games, there are two natural questions: (i) which player has a winning strategy and (ii) what is the minimum number of moves if Dominator has a winning strategy the Maker-Breaker domination number γMB(G) of G is the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. Otherwise we set γMB(G) = ∞. Similarly, γ′MB(G) denotes is the minimum number of moves of Dominator in the game in which Staller plays first. We proceed as follows. In the section(2) we list some definitions and needed results in this paper, as well as prove some basic results on the Maker-Breaker domination number. In Section (3) we first compare γMB(G) with γ′MB(G) and find out that they behave totally diff erent than the related game domination invariants. We also compare γMB(G) with the domination numberand using the Erd″os-Selfridge Criterion prove that if the number of γ-sets of G is not too big, then γMB(G) > γ(G). In Section (4) we introduce residual graphs, determine(resp. bound) γ′MB(G) (resp. γMB(G)) in terms of the residual graph, and determine γMB(T) and γ′MB(T) for an arbitrary tree.

In the next two sections we obtain the invariants for cycles and bound them for union of graphs. We conclude with a list of open problems and directions for further investigation.

2 Preliminaries

Let G be a graph. A vertex of G adjacent to a leaf is a support vertex of G. A perfect matching of G is a set of pairwise independent edges that cover V (G). The order of G will be denoted with n(G). If u is a vertex of G, then N[u]denoted the closed neighbourhood of u.if v is another vertex then we set N[u,v] = N[u]∩N[v]. A set D ⊆ V (G) is a dominating set of G if $U_{u\in D}$N[u] = V (G). The domination number γ(G) is the size of a smallest dominating set of G.

A dominating set of size γ(G) is called a γ-set of G. The Maker-Breaker domination game is called a D-game (resp. S-game) if Dominator (resp. Staller) is the first to play a vertex. The sequence of vertices selected in a D-game willbe denoted with $d_{1,}s_{1,}d_2,s_{2,}...$, and the sequence of vertices selected in an S-game with $s'_1,d'_{1,}s'_2,d'_2$ … … …Suppose that Dominator wins a D-game. Then the last vertex played is by Dominator, let it be d_k. By the definition of the game, $\{d_{1,}d_2,d_3,...,d_k\}$ is a dominating set of G. Similarly, if Dominator wins an S-game and the last vertex played by Dominator is $d'_{l,}$then $\{d'_{1,}d'_{2,} \dots \dots d'_l\}$ is a dominating set ofG . We say that a move of Staller is a double threat if it creates two possibilities for her to win in the next move and consequently Dominator cannot prevent Staller to win. Let G be a graph, k ≥ 1, and $u_{1,}u_{2,} \ldots . u_k, v_1, v_2, \ldots . . v_k$pairwise different vertices ofG . Then we say that X = $\{u_{1,}, v_1\}$,...,$\{u_k, v_k\}\}$ is a pairing dominating set

if$\bigcup_{i=1}^{k}$ N$[u_{i,}, v_i]$= V (G).

Lemma 2.1

Let$u_{1,}u_{2,} \ldots . u_k, v_1, v_2, \ldots . . v_k$be pairwise different vertices of a graph G, and let X = $\{\{u_{1,}, v_1\}$,...,$\{u_k, u_{k,}\}\}$Then X is a pairing dominating set if and only if every set$\{x_1, x_2,...,x_i\}$, where $x_i \in \{u_{i,} v_i\}$, i ∈ [k], is a dominating set of G.

Proof.

Suppose first that X is a pairing dominating set, that is, $\bigcup_{i=1}^{k}$ N$[u_{i,}, v_i]$= V (G). Let $\{x_1, x_2,...,x_k\}$ be an arbitrary set with $x_i \in \{u_{i,} v_i\}$ i ∈ [k]. V (G) = $\bigcup_{i=1}^{k}$ N$[u_{i,}, v_i] \subseteq \bigcup_{i=1}^{k}$ N$[x_i]$so $\{x_1, x_2,...,x_k\}$ is a dominating set of G.

Conversely, consider a set$\{x_1, x_2,...,x_k\}$, where $x_i \in \{u_{i,} v_i\} y_i \in$ [k], and supposethat V (G) = $\bigcup_{i=1}^{k}$ N$[u_{i,}, v_i]$is a proper subset of V (G). Let w ∈ V (G)\$\bigcup_{i=1}^{k}$ N$[u_{i,}, v_i]$. Then for every i ∈ [k] we

have$\mathbf{W} \notin$N$[u_{i,}, v_i]$Let$y_i \in \{[u_{i,}, v_i\}$ be such that$\mathbf{W} \notin$N$[y_i]$. Butthen

$\bigcup_{i=1}^{k}$ N$[y_i]$is a proper subset of V (G), that is, $\{y_1, y_2,...,y_k\}$ is not a

dominatingset.

Lemma 2.2

If G admits a pairing dominating set, then Dominator has a winning strategy on G in the D-game as well as in the S-game.

The converse of Lemma2.2 does not hold in general. For instance, a chordal graph is presented on which Dominator has a winning strategy in both games but admits no pairing dominating set. On the other hand, the converse holds in the class of trees because if Dominator has a winning strategy on a tree T, then it was proved in that T has a dominating matching. Moreover, the converse also holds for co-graphs.

Lemma 2.3 (No-Skip Lemma)

In an optimal strategy of Dominator to achieve γMB(G) or γ′MB(G) it is never an advantage for him to skip a move. Moreover, if Staller skips a move it can only be an advantage for Dominator.

Proof.

Suppose a D-game or an S-game is played. Let Dominator and Staller play optimally until some point when Staller decides to skip a move. In that case, Dominator imagines an arbitrary move of Staller, say x, and replies optimally to this move. Since Dominator can always, no matter the way Staller selects vertices, finish the game in no more that γMB(G) (resp. γ′MB(G)) moves, this property is preserved after the imagined move x and the reply to it. Then Dominator proceeds until the end of the game with the same strategy. Note that it may happen that in the course of the game Staller selects a vertex which is not a legal move in the game Dominator is imagining. In that case Dominator imagines that yet some other legal move has been played by Staller. In this way the game on G will finish in no more than γMB(G) (resp. γ′MB(G)) moves. With a strategy of Staller parallel to the above strategy of Dominator we also infer that it is never an advantage for Dominator to skip a move

Lemma 2.4 (Continuation Principle)

Let G be a graph with A,B⊆ V (G). If B ⊆ A, then γMB(G|A) ≤ γMB(G|B) and γ′MB(G|A) ≤ γ′MB(G|B).

Indeed, the remark follows from the fact that Dominator can apply the same strategy in G|A as in G|B.

Suppose that γMB(G) < ∞. Then in any winning strategy of Dominator, he will play at most half of the vertices (because Staller will play the other half)which in turn implies that $1 \leq \gamma MB(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$ (1)

The bound is sharp, consider for instance the disjoint union of k_1 and several copies of k_2. It is also easy to see that all the possible values from (1) can be realized by considering the disjoint union of a complete graph and an appropriate number of k_2s. Similarly, for the S-game, assuming thatγ′MB(G) < ∞, we have $1 \leq \gamma' MB(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$(2) where again all the values can be realized.

Later we will apply the celebrated Erd″os-Selfridge Criterion for Maker-Breaker games that reads as follows.

Theorem 1 (Erd″os-Selfridge Criterion [13]) If $\mathcal{F}$ is a hypergraph, then $\sum_{A\in\mathcal{F}} 2^{-|A|}<\frac{1}{2}\Rightarrow$ F is a Breaker's win. This theorem together with its proof can also be found in the book [16, Theorem 2.3.3].

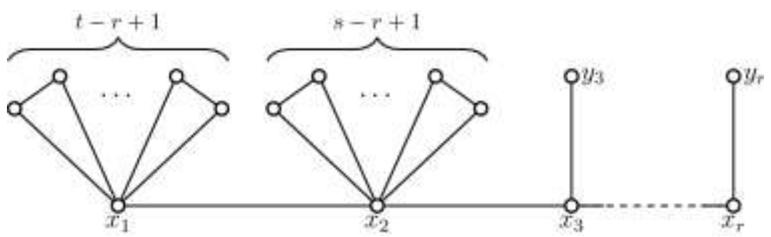
3 Maker-Breaker domination numbers

In this section we first compare γMB(G) with γ′MB(G) and construct graphs for all possible values of the invariants. In the second part we compare γMB(G) with the domination number and using the Erdo″s-Selfridge Criterion find a large class of graphs G for which γMB(G) > γ(G) holds.

3.1 Realizations of Maker-Breaker domination numbers forgraphs

One of the fundamental theorems on the domination game proved in [4, 21] asserts that |γg(G)−γ′g(G)|≤ 1 holds for every graph G. The next result reveals that the situation with the Maker-Breaker domination number is dramatically diff erent. Theorem 3.1 If G is a graph, then γ(G) ≤ γMB(G) ≤ γ′MB(G). Moreover, for any integers r,s,t, where 2 ≤ r ≤ s ≤ t, there exists a graph G such that γ(G) = r, γMB(G) = s, and γ′MB(G) = t.

Example of the graph



Relation with the domination number

As already observed above, γ(G) = 1 if and only if γMB(G) = 1. In general it would be interesting to characterize the graphs G such that γMB(G) = γ(G) = k, where k ≥ 2 is a fixed integer. For k = 2 the answer is simple

Residual graphs

In this section we study the Maker-Breaker domination number on a construction that might be of independent interest and that will be later used to determine the invariant for trees. If G is a graph, then we say that the residual graph R(G) of G is the graph obtained from G by iteratively removing pendant path$p_2$until no such path is present. By a pendant $p_2$we mean$p_2$attached to G with an edge. Hence, when such a pendant$p_2$is removed, exactly two vertices and two edges are removed. When G = p_2, we can also remove it and obtain the empty graph. Note that H = R(G) for some graph G if and only if H is the empty graph, H =k_1, or each support vertex of H has degree at least 3. This is in particular true if H has no support vertices. We further observe:

Result:1

If T a tree, then

$$\gamma MB(G)=\begin{cases} \frac{n(T)}{2}, T \ \ has\ a\ perfect\ matching \\ \frac{n(T)-1}{2}, R(T) \cong k_1 \\ \frac{n(T)-k+1}{2}, R(T) \cong k_{1,k} \\ \infty, otherwise \end{cases}$$

$$\text{And } \gamma' MB(T)=\begin{cases} \frac{n(T)}{2}, T \ \ has\ a\ perfect\ matching \\ \infty, otherwise \end{cases}$$

Result:2

If G admits a pairing dominating set, then Dominator has a winning strategy on G in the D-game as well as in the S-game.

Main Theorem:1

If n ≥ 5, then γMB(C_n) = γ′MB(C_n) = $\left\lfloor\frac{n}{2}\right\rfloor$

Proof.

We begin by showing that γMB(C_n) ≤ $\left\lfloor\frac{n}{2}\right\rfloor$and γ′MB($C_n$) ≤ $\left\lfloor\frac{n}{2}\right\rfloor$. If n iseven, $C_n$has a perfect matching (which is also a dominating matching), thus by result 2, it holds that γMB(C_n) ≤ $\frac{n}{2}$ =$\left\lfloor\frac{n}{2}\right\rfloor$and γ′MB($c_n$) ≤$\left\lfloor\frac{n}{2}\right\rfloor$Now consider thecase when n is odd. In a D-game, let v be the first vertex played by Dominator. Clearly among undominated vertices, V (c_n) − N[v], there is a perfect matching. Thus γMB(C_n) ≤ 1 + $\frac{n-5}{2}$=$\left\lfloor\frac{n}{2}\right\rfloor$In an S-game, suppose s'_1= u. Then Dominator should reply on a neighbour v of the vertex u. Now there is a perfect matching among V (C_n) − N[v], so γ′MB(C_n) ≤ 1 + $\frac{n-5}{2}$=$\left\lfloor\frac{n}{2}\right\rfloor$. This proves the upper bounds. To find the lower bounds we need to find an appropriate strategy for Staller. Set for the rest of the proof that V (C_n) = {x_1,...,x_n}, wherethe adjacencies are natural. We first show the lower bound for the S-game: γ′MB(C_n) ≥$\left\lfloor\frac{n}{2}\right\rfloor$. Suppose, withoutloss of generality, that s'_1= x_1. Notice that Dominator has to reply on a neighbor of x_1, for otherwise Staller plays as $s'_2$the not yet dominated neighbor of x_1. Then Dominator cannot in one move dominate $s'_1$and s'_2. Say he leaves s'_2undominated. Then Staller can play the other neighbor of s'_2 and win the game as Dominator cannot dominates'_2.similiarlywithoutloss of generality, that s'_2= x_3. Notice that Dominator has to reply on a neighbor of x_3, for otherwise Staller plays as $s'_3$the not yet dominated neighbor of x_3. Then Dominator cannot in one move dominate $s'_3$and s'_4Say he leaves s'_4undominated. Then Staller can play the other neighbour of $s'_3$and win the game as Dominator cannot dominate s'_4So without loss of generality, Dominator replies with d'_1=x_n. Staller's next move is s'_3= x_5. In order to prevent Staller from winning, Dominator has to play d'_3= x_4. And Then Staller continues with the same strategy, forcing Dominator to play on (almost) every second move.

If n is even, the game ends after Staller plays x_n−1 and Dominator replies on x_n−2. So in this case, Dominator plays all even labelled vertices, hence γ′MB(C_n) ≥ $\frac{n}{2}$=$\left\lfloor\frac{n}{2}\right\rfloor$.

If n is odd, the game ends after Staller plays x_n−2 and Dominator replies on x_n−3 (as x_n−1 is already dominated by d'_1). So Dominator again plays all even labelled vertices, thus γ′MB(C_n) ≥ $\frac{n-1}{2}$=$\left\lfloor\frac{n}{2}\right\rfloor$.

It remains to prove that γMB(C_n) ≥$\left\lfloor\frac{n}{2}\right\rfloor$. Assume without loss of generality that d_1= x_1. Staller replies on s_1 =x_4 (so at distance 3 from d_1). Again, Dominator has to reply on a neighbour of s_1. If he replies on x_3, then Staller can apply the above strategy by playing x_6 next, and then every second vertex along the cycle to ensure at leastn 2moves. If Dominator replies on d_2= x_5, then Staller plays s_2 =x_8 (again at distance 3 from d_2 in the same direction as before). But observe that at some point of the game Dominator will have to play on {x_2,x_3} to dominate the whole graph. By repeating this strategy, Staller ensures that among every four consecutive vertices of the cycle, Dominator makes at least two moves (except maybe in the last one, two or three remaining vertices). We now distinguish four diff erent cases.

• If n ≡ 0 mod 4, then no vertex remains and γMB(Cn) ≥$\frac{2n}{4}$=$\left\lfloor\frac{n}{2}\right\rfloor$.

• If n ≡ 1 mod 4, then only one vertex remains, which is already dominated by d1, so γMB(Cn) ≥$\frac{2(n-1)}{4}$=$\left\lfloor\frac{n}{2}\right\rfloor$.

• If n ≡ 2 mod 4, then among the remaining two vertices, one is dominated by d_1 but the other is not. So Dominator has to make another move, thus

γMB(Cn) ≥ $\frac{2(n-2)}{4}+1$=$\left\lfloor\frac{n}{2}\right\rfloor$

• If n ≡ 3 mod 4, then two of the remaining three vertices are not yet dominated,

so Dominator still has to make just one more move. So γMB(Cn) ≥$\frac{2(n-3)}{4}$ = $\left\lfloor\frac{n}{2}\right\rfloor$.

• If n ≡4 mod 4, then the remaining four vertices two is dominated by d1 and $d_2$but the other is not. So Dominator has to make another move, thusγMB(Cn) ≥$\frac{2(n-4)}{4}$+2 = $\left\lfloor\frac{n}{2}\right\rfloor$.

• If n ≡5 mod 4, then two of the remaining five vertices are not yet dominated thusγMB(Cn)≥$\frac{2(n-5)}{4}$ = $\left\lfloor\frac{n}{2}\right\rfloor$.

Conclusion

To conclude the paper we list several problems and directions for further investigation of the Maker-Breaker domination number

In this paper we have considered the Maker-Breaker domination number which is an optimization problem from Dominator's point of view. It would likewise be of interest to consider the Staller's point of view, that is, assuming that Staller wins on a graph G, what is the minimum number of moves with which she can achieve the goal?

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