Complementary Tree Domination Number of Semitotal Point Graph

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Abstract—A set \( D \subseteq V \) of a graph \( G = (V, E) \) is a complementary tree dominating set if the induced subgraph \( <V-D> \) is a tree. The complementary tree domination number \( \gamma_{ctd}(G) \) is the minimum cardinality of a complementary tree dominating set (ctd-set) of \( G \). The semitotal-point graph \( T_2(G) \) is the graph \( G \) whose vertex set is \( V(G) \cup E(G) \). Where two vertices are adjacent if and only if (i) they are adjacent vertices of \( G \) or (ii) one is a vertex and the other is an edge of \( G \) incident with it. In this paper complementary tree domination number for semitotal-point graph on some standard graphs and bounds are obtained.

Keywords—Dominating set, Complementary tree dominating set, Semi total point graph.

I. INTRODUCTION

E. SampathKumar and S.B. Chikkodimath introduced the concept of Semi Total-point graph of a graph [5]. Graphs discussed in this paper are simple and undirected graphs. Complementary tree domination number of a graph was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya [3]. They defined the complementary tree dominating set, the complementary tree domination number are obtained for some standard graphs. In this paper, exact values of some standard graphs and bounds of \( \gamma_{ctd}(T_2(G)) \) are found.

Definition 1.1 [3]

A set \( D \subseteq V(G) \) is said to be complementary tree dominating set (ctd-set) if the induced subgraph \( <V(G) - D> \) is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of \( G \) and is denoted by \( \gamma_{ctd}(G) \).

Definition 1.2 [1]

The semi total-point graph \( T_2(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \), where two vertices are adjacent if and only if (i) they are adjacent vertices of \( G \) or (ii) one is a vertex of \( G \) and the other is an edge of \( G \) incident with it.

Example:
II. PRIOR RESULTS

Proposition 2.1[3]
Pendant vertices are members of every ctd-set.

Theorem 2.2[3]
For any connected graph G with \( p \geq 2 \), \( \gamma_{ctd}(G) \leq p - 1 \).

Observation 2.3[3]
(i) For any path \( P_n \) with \( n \geq 4 \) vertices, \( \gamma_{ctd}(P_n) = n - 2 \).
(ii) For any cycle \( C_n \), \( \gamma_{ctd}(C_n) = n - 2 \) if \( n \geq 3 \).
(iii) For any star \( K_{1,n} \), \( \gamma_{ctd}(K_{1,n}) = n \) if \( n \geq 2 \).
(iv) For any wheel \( W_p \) with \( p \geq 4 \) vertices, \( \gamma_{ctd}(W_p) = 2 \).
(v) For any complete graph \( K_n \) with \( n \geq 3 \) vertices, \( \gamma_{ctd}(K_n) = n - 2 \).

III. COMPLEMENTARY TREE DOMINATION NUMBER FOR SEMITOTAL-POINT GRAPH \( T_2(G) \) OF SOME STANDARD GRAPHS

In the following, exact values of complementary tree domination number of semitotal-point graph of some classes of graphs are given.

Proposition 3.1
For any path \( P_p \) with \( p \geq 2 \) vertices, \( \gamma_{ctd}(T_2(P_p)) = p - 1 \).

Proof.
Let \( P_p \) (\( p \geq 2 \)) be a path with vertex set \( \{v_1, v_2, \ldots, v_p\} \) and edge set \( \{e_1, e_2, \ldots, e_{p-1}\} \). For Semitotal-point graph \( T_2(P_p) \) whose vertex set is \( V(G) \cup E(G) \) so that \( \{e_1, e_2, \ldots, e_{p-1}\} \) is labelled as vertex set \( \{u_1, u_2, \ldots, u_{p-1}\} \) then \( V(T_2(P_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_{p-1}\} \). Let D be the ctd-set of \( T_2(P_p) \). So D = \( \{u_1, u_2, \ldots, u_{p-1}\} \) then \( |D| = p - 1 \) Hence \( \gamma_{ctd}(T_2(P_p)) = p - 1 \).

Proposition 3.2
For any cycle \( C_p \) with \( p \geq 3 \) vertices, \( \gamma_{ctd}(T_2(C_p)) = p - 1 \).

Proof.
Let \( C_p \) (\( p \geq 3 \)) be a cycle with vertex set \( \{v_1, v_2, \ldots, v_p\} \) and edge set \( \{e_1, e_2, \ldots, e_p\} \). Let \( u_1, u_2, \ldots, u_p \) be the vertices corresponding to the edges of \( C_p \) to obtain \( T_2(C_p) \). Then \( V(T_2(C_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_p\} \). Let \( D = \{v_1, v_2, \ldots, u_{p-1}\} \) be a ctd-set of \( T_2(C_p) \). Therefore \( |D| = p - 1 \). Hence \( \gamma_{ctd}(T_2(C_p)) = p - 1 \).

Proposition 3.3
For a star graph \( K_{1,p-1} \), \( p \geq 2 \) then \( \gamma_{ctd}(T_2(K_{1,p-1})) = p - 1 \) or \( q \).

Proof.
Let \( K_{1,p-1} \) be a star graph with \( p \) vertices say \( v, v_1, v_2, \ldots, v_{p-1} \) and edge set is \( \{e_1, e_2, \ldots, e_{p-1}\} \). Then the \( V(T_2(K_{1,p-1})) = \{v, v_1, v_2, \ldots, v_{p-1}, u_1, u_2, \ldots, u_{p-1}\} \). Each \( u_i \) (\( 1 \leq i \leq p-1 \)) dominates \( \{v, v_i\} \) (\( 1 \leq i \leq p-1 \)) So that \( D = \{u_i \mid 1 \leq i \leq p-1\} \) is a ctd-set of \( T_2(K_{1,p-1}) \).

Hence \( \gamma_{ctd}(T_2(K_{1,p-1})) = p - 1 \).

Proposition 3.4
For a complete graph \( K_p \), \( p \geq 3 \) then \( \gamma_{ctd}(T_2(K_p)) = \frac{p^2 - 3p + 4}{2} \).

Proof.
Let \( K_p \) be a complete graph with \( p \geq 3 \) vertices. Let \( \{v_1, v_2, \ldots, v_p\} \) be the vertex set and \( \{(v_1,v_2),(v_1,v_3),\ldots,(v_1,v_p),(v_2,v_3),\ldots,(v_{p-1},v_p)\} \) is an edge set is denoted by vertex set \( \{u_1, u_2, \ldots, u_{p-1}, u_{p-1}p\} \) in \( T_2(K_p) \) where
\[ T_2(K_p) = V(K_p) \cup E(K_p) \] and \[ |V(T_2[K_p])| = p + \frac{p(p-1)}{2} \]. we know \( \gamma_{ctd}(K_p) = p - 2 \) [2.3(v)] so that choose the vertices \( v_1, v_2, \ldots, v_{p-2} \), also choose the corresponding edge of \( G \) as vertices \( \{u_i, 1 \leq i \leq p - 2, i + 1 \leq j \leq p - 2\} \) Since \( \{v_p, v_{p+1}\} u_{p+1, p} \) forms a cycle, so that
\[
D = \{u_i, 1 \leq i \leq p - 1\} \cup \{v_j, 1 \leq i \leq p - 2, i + 1 \leq j \leq p - 2\} \cup \{u_{p+1, p}\}
\]
\[
|D| = (p-2)(p-1) + 1
\]
\[
= \frac{p^2 - 3p + 4}{2}
\]
Hence \( \gamma_{ctd}(T_2[K_p]) = \frac{p^2 - 3p + 4}{2} \).

**Proposition 3.5**

For any wheel \( W_p \) with \( p \) vertices, \( \gamma_{ctd}(T_2[W_p]) = 2(p-1) \).

**Proof.**

Let \( v_0, v_1, \ldots, v_{p-1} \) be the vertices of wheel \( W_p \), where \( d(v_0) = p - 1 \) and edge set is \( \{(v_0, v_1), (v_1, v_2), \ldots, (v_{p-2}, v_{p-1}), (v_{p-1}, v_0)\} \). The semitotal-point graph \( T_2(T) \) is the graph whose vertex set is \( V(T) \cup E(T) \) where \( V(T_2(T)) = \{v_1, v_2, \ldots, v_p, u_{i1}, u_{i2}, \ldots, u_{i(p-1)}\} \). Let \( D = \{u_i, 1 \leq i \leq p - 1\} \) is a minimal ctd-set of \( T \). Hence \( \gamma_{ctd}(T_2[T]) = p - 1 \).

**Observation 3.6**

For any tree \( T \) of \( p \geq 2 \) vertices then \( \gamma_{ctd}(T_2(T)) = p - 1 \).

**Proof.**

Let \( T \) be a tree with \( p \) vertices \( v_1, v_2, \ldots, v_p \) and edges \( e_1, e_2, \ldots, e_{p-1} \). The semitotal-point graph \( T_2(T) \) is the graph whose vertex set is \( V(T) \cup E(T) \) where \( V(T_2(T)) = \{v_1, v_2, \ldots, v_p, u_{i1}, u_{i2}, \ldots, u_{i(p-1)}\} \). Let \( D = \{u_i, 1 \leq i \leq p - 1\} \) is a minimal ctd-set of \( T \). Hence \( \gamma_{ctd}(T_2[T]) = p - 1 \).

**IV. BOUNDS OF \( \gamma_{ctd}(T_2(G)) \)**

**Theorem 4.1**

At least one edge of \( G \) is the member of ctd-set of \( T_2(G) \).

**Proof.**

Let \( G(p,q) \) be a connected graph. Then \( |V(T_2(G))| = p + q \) and \( V(T_2(G)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\} \) each \( u_i \) is adjacent with two vertices of \( v_i \) which forms a cycle. Suppose choose a vertex \( v_i \in D \) then \( V(D) \) is disconnected. So that \( u_i \in D \) therefore \( D \) contains at least one edge of \( G \). Hence the theorem.

**Theorem 4.2**

For any \( (p,q) \) graph \( G \), \( \gamma_{ctd}(G) \leq \gamma_{ctd}(T_2(G)) \), the equality holds for \( G = K_2 \).

**Proof.**

Let \( G \) be a graph, then by theorem 2.2, \( \gamma_{ctd}(G) \leq p - 1 \). Therefore by definition of \( T_2(G) \), \( \gamma_{ctd}(G) \leq p + q - 2 \) Hence \( \gamma_{ctd}(G) \leq \gamma_{ctd}(T_2(G)) \).

For equality can be easily verified.
Theorem 4.3
For any connected graph $G$ with $p \geq 2,1 \leq \gamma_{ctd}(T_2(G)) \leq p + q - 2$.

Proof.
$T_2(G)$ has $p + q$ vertices and radius of $T_2(G)$ is one. Hence $\gamma_{ctd}(T_2(G)) \geq 1$. Also there is no vertex of degree $p + q - 1$ in $T_2(G)$, so that $|D| \leq p + q - 2$. Therefore $1 \leq \gamma_{ctd}(T_2(G)) \leq p + q - 2$.

Theorem 4.4
For any non trivial tree $T, \gamma_{ctd}(T) \leq \gamma_{ctd}(T_2(T))$.

Proof.
Let $T$ be any non trivial tree. Let $v_1, v_2, \ldots, v_n$ be a subset of the vertex set such that $\deg(v_i)$ for $1 \leq i \leq m$ and $u_1, u_2, \ldots, u_n$ be the subset of $V(T)$ where $\deg(u_i) \geq 2$ and also a path of length $n$. So that $n + m = p$. We know pendant vertices are members of every ctd-set $[2.1]$ and ctd-set is of path of length $n$ is $n - 2$ [2.3(i)]. Let $D$ be the ctd-set of $T$. Therefore $|D| \leq m + n - 2$ therefore $\gamma_{ctd}(T) \leq m + n - 2$.

Corollary 4.5
In a tree if $\Delta(T) = p - 1$ then $\gamma_{ctd}(T_2(T)) = \gamma_{ctd}(T)$.

Theorem 4.6
$\gamma_{ctd}(T_2(G)) = 1$ if and only if $G \cong K_2$.

Proof.
Let $D = \{u\}$ be a ctd-set of $T_2(G)$ such that $|D| = 1$ and $\langle V - D \rangle \cong T$. Where $u \in V(T_2(G))$ By theorem (4.1) $u$ is adjacent to two vertices of $G$ which is connected. Hence $G \cong K_2$. Conversely, if $G \cong K_2$ then $\gamma_{ctd}(T_2(G)) = 1$.

REFERENCES