# A STUDY ON PARTIAL SURFACE LANGUAGES

Anitta Arulraj*[1], Robinson Thamburaj*[2]
*Department of Mathematics, Madras Christian College, Chennai,*
[1]anittaarul@gmail.com
[2]robinson@mcc.edu.in

***Abstract*— In the area of combinatorial studies of string languages under formal grammars, the concept of Generalized Parikh Vector (GPV) gives the positions of symbols in linear strings. It has been proved that GPVs of the strings of the same length lie on a hyper plane. The studies on GPV of strings over a binary alphabet gave rise to the concept of line languages. The concept has been extended to Partial surface languages and Curve languages in this paper.**

***Keywords*— Generalized Parikh Vector, Partial Surface Languages, Partial Spherical Languages, Partial Curve Languages, Language Curve**

## I. Introduction

The theory of formal languages is a fundamental area of Theoretical Computer Science having its origin in the study of Noam Chomsky in the 1950's on grammars and grammatical structure of a language with extensive research being done for more than five decades. In theory of formal languages, combinatorial aspects of words and languages have been of great interest. Words (strings of symbols) are fundamental in computer processing activities. Combinatorics of Words is branching from Combinatorics, deals with the study of words. Berstel and Boasson defined a partial words in context of protein comparison in which words from a finite alphabet may have a "do not know" symbol denoted by .Parikh vector and Generalized Parikh Vector are two important tool in the combinatorial studies of words.

In section 1 - the definitions of parikh vector, generalized parikh vector, line languages, surface languages and curve languages. In section 2 - talks about quasi partial Spherical language. In section 3 - about the quasi partial spherical languages. In section 4 - discuss about the condition for a quasi-partial curve languages and Section 5 talks about the closure properties of a quasi-partial curve languages.

## II. PRELIMINARIES

### A. *Parikh Vector*

Let $\sum=\{a,b,c\}$ then the Parikh Vector of a word u is given by $\pi(x)=\left(|u|_{a_1},|u|_{a_2},\ldots|u|_{a_n}\right)$ where $|u|_{a_n}$ represents the number of times $a_n$ occur in u.

- Let $\sum=\{a,b,c\}$ , if u=abbcabac, then the Parikh Vector of a word u is given by $\pi(x)=(3,3,2)$.

### B. *Generalised Parikh Vector*

Let $\sum=\{a,b,c\}$ and $x\in\sum^{\infty}$. The Generalized Parikh Vector (GPV) of x denoted by p(x) is $p(x)=(p_1,p_2,p_3)\in[0,1]^3$ where $p_i=\sum_{j\in A_i}\frac{1}{2^j}$,

$A_i\subset N$ (set of natural numbers) and $A_i$ contains the position of $a_i$ in the word x.

Let $x=abababc\in\sum^{\infty}$, then $p(x)=\left(\frac{1}{2^1}+\frac{1}{2^3}+\frac{1}{2^5},\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6},\frac{1}{2^7}\right)$.

### C. *Point Languages*

A language $L\in\sum^{\infty}$ if $|L|=1$ then L is called a Point language.

*D. Partial Line Languages*

For $\Sigma$ = {a, b}, a language $PL \subset \sum^{\infty}$ is called a Partial line language if there exists a line l in $R^2$ such that PL ={ $x \in \sum^{\infty}$:p(x) lies on l, x is a partial word}. Then, a line l is said to be the Partial language line of PL.

If PL contains only finite words then PL is a finite Partial line language, if PL contains only infinite words then PL is an $\omega$ - Partial line language and if PL contains both finite and infinite words then PL is an infinitary Partial line language.

Let $PL = \{\diamond\diamond a, \diamond\diamond b\}$. PL is finite Partial line language and its corresponding finite Partial Language line is $x + y = \frac{1}{8}$

*E. Partial Surface Languages*

For Σ = {a, b, c}, $PL \subset \sum^{\infty}$ is a Partial language if there exists a surface S such that PL = { $x \in \sum^{\infty}$: p(x) lies on a surface S, where x is a partial words} then PL is called a Partial surface languages and S is said to be a Partial Language surface. S may be a plane, sphere etc.

$PL \subset \sum^{\infty}$ is a partial language and if there exists a surface $s: (x - k_1)^2 + (y - k_2)^2 + (z - k_3)^2 = r^2$ in $R^3$ such that PL = { $x \in \sum^{\infty}$: p(x) lies on the non-planar surface S, where x is a partial words} then PL is called a Partial spherical language and S is said to be a Partial Language surface.

*F. Quasi-Partial Surface Languages*

For Σ = {a, b, c}, $PL \subset \sum^{\infty}$ is a Partial language and if there exists a surface $Q: (x - k_1)^2 + (y - k_2)^2 + (z - k_3)^2 = r^2$ in $R^3$ such that PL = { $x \in \sum^{\infty}$: p (PL) lies on the non-planar surface Q, where x is a partial words} then PL is a Quasi Partial Spherical language and Q is a Quasi Partial Language surface.

## III. Some Observations on Line and Plane languages

The following are some observation on Line and Plane languages:

*A. Theorem 3.1*

For an Ellipsoid $S: \frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} + \frac{z^2}{k_3^2} = 1$

- If $k_1 = k_2 \neq k_3$ or $k_1 \neq k_2 = k_3$ or $k_1 \neq k_3 = k_2$. Then $|C_{Gpv}| = 2$ (i.e.) S is a finite Language line and its corresponding language is a finite line language.
- If $k_1 \neq k_2 \neq k_3$ then $|C_{Gpv}| = 1$ (i.e.) S is a Language point and its corresponding language is a point language

where $x \neq y \neq z$, x, y, z are of the form $\frac{1}{2^i}, i > 0, k_1, k_2, k_3 \in R$

Example 1 Let L={abc,bac}.L is a finite Line language and its corresponding finite language line is $\frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} + \frac{z^2}{k_3^2} = 1$ where

$k_1^2 = k_2^2 = \frac{3}{2^3}, k_3^2 = \frac{3}{2^4}$

Example 2 Let L={abc}.L is a Point language and its corresponding language point is $\frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} + \frac{z^2}{k_3^2} = 1$ where

$k_1 = \frac{3}{2}, k_2 = \frac{3}{2^4}, k_3 = \frac{1}{2^5}$

*B. Theorem 3.2*

For an Hyperbola $S: \frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} - \frac{z^2}{k_3^2} = 1$

- If $k_1 = k_2 \neq k_3$ or $k_1 \neq k_2 = k_3$ or $k_1 \neq k_3 = k_2$. Then $|C_{Gpv}| = 2$ (i.e.) S is a finite Language line and its corresponding language is a finite line language.
- If $k_1 \neq k_2 \neq k_3$ then $|C_{Gpv}| = 1$ (i.e.) S is a Language point and its corresponding language is a point language

where $x \neq y \neq z, x, y, z$ are of the form $\frac{1}{2^i}, i > 0, k_1, k_2, k_3 \in R$

Example 1 Let L={abc,bac}.L is a finite Line language and its corresponding finite language line is $\frac{x^2}{k_1^2}+\frac{y^2}{k_2^2}-\frac{z^2}{k_3^2}=1$ where

$k_1^2 = k_2^2 = \frac{1}{2^2}, k_3^2 = \frac{1}{2^4}$

Example 2 Let L={abc}.L is a Point language and its corresponding language point is $\frac{x^2}{k_1^2}+\frac{y^2}{k_2^2}+\frac{z^2}{k_3^2}=1$ where

$k_1 = \frac{1}{2^2}, k_2 = \frac{1}{2^4}, k_3 = \frac{1}{2^6}$

*C. Theorem 3.3*

For an Cylinder $S:(x-k_1)^2+(y-k_2)^2=r^2$

- If $k_1 = k_2$ or $k_1 = k, k_2 = 0$ or $k_1 = 0, k_2 = k$ .Then $|C_{Gpv}| = 2$ (i.e.) S is a finite Language line and its corresponding language is a finite line language.
- If $k_1 \neq k_2$ then $|C_{Gpv}| = 1$(i.e.) S is a Language point and its corresponding language is a point language

where $x \neq y \neq z, x, y, z$ are of the form $\frac{1}{2^i}, i > 0, k_1, k_2 = \frac{n-1}{2^{i-1}}, n \in N$

Example 1 Let L={aaa,bbb}.L is a finite Line language and its corresponding finite language line is $S:(x-k_1)^2+(y-k_2)^2=r^2$ where $k_1^2 = k_2^2 = \frac{1}{2^1}, r^2 = \frac{9}{2^6}$

*D. Theorem 3.4*

For a sphere $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$

- If $k_1 = k_2 \neq k_3$ or $k_1 \neq k_2 = k_3$ or $k_1 \neq k_3 = k_2$
  Then $|C_{Gpv}| = 2$ (i.e.) S is a finite Language line and its corresponding language is a finite line language.
- If $k_1 \neq k_2 \neq k_3$ then $|C_{Gpv}| = 1$(i.e.) S is a Language point and its corresponding language is a point language
- If $k_1 = k_2 = k_3$ then either $|C_{Gpv}| = 6$ or $|C_{Gpv}| \geq 6$ (i.e.) S is either Language plane or Language surface.

where $x \neq y \neq z, x, y, z$ are of the form $\frac{1}{2^i}, i > 0, a, b, c \in R$

Example: Let $L = \{abc, acb, bac, bca, cab, cba\}$ ,L is a plane language and its Corresponding language plane is $(x-1)^2+(y-1)^2+(z-1)^2 = \frac{101}{2^6}$

Example: Let $L = \{abac, acab, babc, bcba, cacb, cbca, abba, acca, baab, bccb, caac, cbbc\}$ Then L is a surface language and its corresponding language surface $x^2+y^2+z^2 = \frac{117}{2^8}$.

## IV. FINITE QUASI-PARTIAL SPHERICAL LANGUAGE AND FINITE QUASI-PARTIAL LANGUAGE SPHERICAL

The following are some observation about Gpv's of partial words in finite quasi partial Spherical languages over three letter alphabet:

- Words $\begin{pmatrix} \Diamond^{p_1-(2n+1)-1}(ab)^n ba, \Diamond^{p_1-(2n+1)-1}(ac)^n ca, \Diamond^{p_1-(2n+1)-1}(ba)^n ab, \Diamond^{p_1-(2n+1)-1}(ca)^n ac, \Diamond^{p_1-(2n+1)-1}(cb)^n bc, \Diamond^{p_1-(2n+1)-1}(bc)^n cb, \\ \Diamond^{p_1-(2n+1)-1}(ab)^n ac, \Diamond^{p_1-(2n+1)-1}(ac)^n ab, \Diamond^{p_1-(2n+1)-1}(ba)^n bc, \Diamond^{p_1-(2n+1)-1}(ca)^n cb, \Diamond^{p_1-(2n+1)-1}(cb)^n ca, \Diamond^{p_1-(2n+1)-1}(bc)^n ba \end{pmatrix}$ of length

  $p_1$ over $\Sigma = \{a, b, c\}$ lie on a sphere $(x-k)^2+(y-k)^2+(z-k)^2 = \left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{p_1}}+k\right)^2 + \left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{p_1-1}}+k\right)^2 + k^2$ ……..….(1)

  where $p_1 > 2n+2$

- The number of words of length $p_1$ over $\sum=\{a,b,c\}$ lying on (1) is 12.
- Word $\begin{pmatrix} \Diamond^{p_1-(2n+1)-1}(ab)^n ca, \Diamond^{p_1-(2n+1)-1}(ac)^n ba, \Diamond^{p_1-(2n+1)-1}(ba)^n cb, \Diamond^{p_1-(2n+1)-1}(bc)^n ab, \Diamond^{p_1-(2n+1)-1}(cb)^n ac, \Diamond^{p_1-(2n+1)-1}(ca)^n bc, \\ \Diamond^{p_1-(2n+1)-1}(ba)^n ac, \Diamond^{p_1-(2n+1)-1}(ca)^n ab, \Diamond^{p_1-(2n+1)-1}(ab)^n bc, \Diamond^{p_1-(2n+1)-1}(cb)^n ba, \Diamond^{p_1-(2n+1)-1}(ac)^n cb, \Diamond^{p_1-(2n+1)-1}(bc)^n ca \end{pmatrix}$,of length $p_1$ over $\sum=\{a,b,c\}$ lie on a sphere $(x-k)^2+(y-k)^2+(z-k)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$ ……..….(2)

  where $p_1 > 2n+2$
- The number of words of length $p_1$ over $\sum=\{a,b,c\}$ lying on (2) is 12.
- The number of words of length $p_2$ over $\sum=\{a,b,c\}$ lying on the sphere $x^2+y^2+z^2=\left(\frac{1}{2^{P_1}}+\frac{1}{2^{P_2}}\right)^2$ either $P_1>P_2, P_1-P_2\geq 2n+1$ or $P_2>P_1, P_2-P_1\geq 2n+1$ where $n, P_1, P_2 \in N$ is 9.

A. *Theorem 4.1*

PL is a finite Quasi-Partial spherical language of a finite partial quasi language spherical $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$ Iff for $k_1,k_2,k_3=k$ and $r^2$ is either $(x-k)^2+(y-k)^2+(z-k)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{P_1-1}}+k\right)^2+k^2$ or $(x-k)^2+(y-k)^2+(z-k)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$ where $p_1 > 2n+2$ or for $k_1,k_2,k_3=0$ and $r^2=\left(\frac{1}{2^{P_1}}+\frac{1}{2^{P_2}}\right)^2$ either $P_1>P_2, P_1-P_2\geq 2n+1$ or $P_2>P_1, P_2-P_1\geq 2n+1$ where $n,P_1,P_2\in N$

Proof:

Let PL is a finite Quasi-Partial spherical language of a finite quasi-partial language spherical $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$

(i.e.) PL = { $x\in\sum^\infty$: p(PL) lies on the non-planar surface $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$ , where x is a partial words}

By theorem 3.4, centre $(k_1,k_2,k_3)=(k,k,k)$

Since $A=\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ba, \Diamond^{p_1-(2n+1)-1}(ac)^n ca, \Diamond^{p_1-(2n+1)-1}(ba)^n ab, \Diamond^{p_1-(2n+1)-1}(ca)^n ac\right)$

$B=\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ac, \Diamond^{p_1-(2n+1)-1}(ac)^n ab, \Diamond^{p_1-(2n+1)-1}(ba)^n bc, \Diamond^{p_1-(2n+1)-1}(ca)^n cb\right)$ and

$C=\left(\Diamond^{p_2-1}a\Diamond^{p_1-p_2-1}a, \Diamond^{p_2-1}b\Diamond^{p_1-p_2-1}b, \Diamond^{p_2-1}c\Diamond^{p_1-p_2-1}c, \Diamond^{p_2-1}a\Diamond^{p_3-p_2-1}c\Diamond^{p_1-p_3-1}b\right)$ where $p_3=p_1-p_2$ are four non planar points. A and B lies on the non-planar surface $s_1:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r_1^2, s_2:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r_2^2$ and $s_3:(x)^2+(y)^2+(z)^2=r_3^2$ Where $r_1^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{P_1-1}}+k\right)^2+k^2, r_2^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$, $r_3^2=\left(\frac{1}{2^{P_1}}+\frac{1}{2^{P_2}}\right)^2$

If $p_1 \leq 2n+2$ , contradict the PL hence $p_1 > 2n+2$

If $P_2-P_1<2n+1$, then it will contradict to the definition of GPV's hence $P_2-P_1\geq 2n+1$.

Conversely, if the centre of the sphere s is $(k_1,k_2,k_3)=(k,k,k)$ then $r^2$ is either $\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{P_1-1}}+k\right)^2+k^2$ or $\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$ where $p_1 > 2n+2$

If the centre of the sphere s is $(k_1,k_2,k_3)=(0,0,0)$ then $r^2=\left(\frac{1}{2^{P_1}}+\frac{1}{2^{P_2}}\right)^2$ where either $P_1>P_2, P_1-P_2\geq 2n+1$ or

$P_2>P_1, P_2-P_1\geq 2n+1$ ($n,P_1,P_2\in N$)

Since $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$ If $s_1:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{P_1-1}}+k\right)^2+k^2$ then

$A=\lozenge^{p_1-(2n+1)-1}(ab)^n ba, \lozenge^{p_1-(2n+1)-1}(ac)^n ca, \lozenge^{p_1-(2n+1)-1}(ba)^n ab, \lozenge^{p_1-(2n+1)-1}(ca)^n ac, \lozenge^{p_1-(2n+1)-1}(cb)^n bc, \lozenge^{p_1-(2n+1)-1}(bc)^n cb$ are planar points lies in $S_1$ .

Since $\left(\sum_{i=1}^{n}\frac{2^{2i}+1}{2^{P_1-1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}+1}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1}}+k\right)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-1}+1}{2^{P_1-1}}+k\right)^2+k^2$ then

$$B=\begin{pmatrix}\lozenge^{p_1-(2n+1)-1}(ab)^n ba, \lozenge^{p_1-(2n+1)-1}(ac)^n ca, \lozenge^{p_1-(2n+1)-1}(ba)^n ab, \lozenge^{p_1-(2n+1)-1}(ca)^n ac, \lozenge^{p_1-(2n+1)-1}(cb)^n bc, \lozenge^{p_1-(2n+1)-1}(bc)^n cb,\\ \lozenge^{p_1-(2n+1)-1}(ab)^n ac, \lozenge^{p_1-(2n+1)-1}(ac)^n ab, \lozenge^{p_1-(2n+1)-1}(ba)^n bc, \lozenge^{p_1-(2n+1)-1}(ca)^n cb, \lozenge^{p_1-(2n+1)-1}(cb)^n ca, \lozenge^{p_1-(2n+1)-1}(bc)^n ba\end{pmatrix}$$ are non-planar points lies in $S_1$.

If $s_2:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$ and

$C=\lozenge^{p_1-(2n+1)-1}(ab)^n ac, \lozenge^{p_1-(2n+1)-1}(ac)^n ab, \lozenge^{p_1-(2n+1)-1}(ba)^n bc, \lozenge^{p_1-(2n+1)-1}(ca)^n cb, \lozenge^{p_1-(2n+1)-1}(cb)^n ca, \lozenge^{p_1-(2n+1)-1}(bc)^n ba$ are planar points lies in $S_2$ .

Since $\left(\sum_{i=1}^{n}\frac{2^{2i}+1}{2^{P_1-1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}+1}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1}}k\right)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}+1}{2^{P_1}}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i-2}}{2^{P_1-2}}+k\right)^2+\left(\frac{1}{2^{P_1-1}}+k\right)^2$ then

$$D=\begin{pmatrix}\lozenge^{p_1-(2n+1)-1}(ab)^n ba, \lozenge^{p_1-(2n+1)-1}(ac)^n ca, \lozenge^{p_1-(2n+1)-1}(ba)^n ab, \lozenge^{p_1-(2n+1)-1}(ca)^n ac, \lozenge^{p_1-(2n+1)-1}(cb)^n bc, \lozenge^{p_1-(2n+1)-1}(bc)^n cb,\\ \lozenge^{p_1-(2n+1)-1}(ab)^n ac, \lozenge^{p_1-(2n+1)-1}(ac)^n ab, \lozenge^{p_1-(2n+1)-1}(ba)^n bc, \lozenge^{p_1-(2n+1)-1}(ca)^n cb, \lozenge^{p_1-(2n+1)-1}(cb)^n ca, \lozenge^{p_1-(2n+1)-1}(bc)^n ba\end{pmatrix}$$ are non-planar points lies in $S_2$.

If $s_3:(x)^2+(y)^2+(z)^2=\left(\frac{1}{2^{P_1}}+\frac{1}{2^{P_2}}\right)^2$, $E=\{\lozenge^{P_2-1} a \lozenge^{P_1-P_2-1} a, \lozenge^{P_2-1} b \lozenge^{P_1-P_2-1} b, \lozenge^{P_2-1} c \lozenge^{P_1-P_2-1} c, \lozenge^{P_2-1} a \lozenge^{P_3-P_2-1} c \lozenge^{P_1-P_3-1} b\}$ are non-planar points lies in $s_3$. If PL is a Partial language consist of words of B or D or E then PL is a finite partial-quasi spherical language of a finite partial quasi language spherical S. Hence the theorem.

*B. Example 4.1.1*

Let $PL=\{\lozenge$ ababba$\lozenge$ acacca ,$\lozenge$ babaab,$\lozenge$ cacaac,$\lozenge$ cbcbbc,$\lozenge$ bcbccb, $\lozenge$ ababac,$\lozenge$ acacab,$\lozenge$ bababc,$\lozenge$ cacacb,$\lozenge$ cbcbca,$\lozenge$ bcbcba$\}$

PL is a finite quasi partial spherical language and its corresponding finite quasi partial language spherical is $x^2+y^2+z^2=\frac{2165}{2^{14}}$

*C. Example 4.1.2*

Let PL = ◊ababba,◊acacca,◊babaab,◊cacaac, ◊cbcbbc,◊bcbccb, ◊ababac,◊acacab,◊bababc,◊cacacb,◊cbcbca,◊bcbcba

PL is a finite quasi partial Spherical language and its finite quasi partial language spherical is

$(x+1)^2+(y+1)^2+(z+1)^2=\frac{67445}{2^{14}}$

## V. $\omega$-QUASI-PARTIAL SPHERICAL LANGUAGE AND $\omega$-QUASI-PARTIAL LANGUAGE SPHERICAL

The following are some observation about Gpv's of partial words in $\omega$-quasi partial Spherical languages over three letter alphabet:

- Words $\begin{pmatrix} (\Diamond^{p_1-(2n+1)-1}(ab)^n ba)^\omega, (\Diamond^{p_1-(2n+1)-1}(ac)^n ca)^\omega, (\Diamond^{p_1-(2n+1)-1}(ba)^n ab)^\omega, (\Diamond^{p_1-(2n+1)-1}(ca)^n ac)^\omega, \\ (\Diamond^{p_1-(2n+1)-1}(cb)^n bc)^\omega, (\Diamond^{p_1-(2n+1)-1}(bc)^n cb)^\omega, (\Diamond^{p_1-(2n+1)-1}(ab)^n ac)^\omega, (\Diamond^{p_1-(2n+1)-1}(ac)^n ab)^\omega, \\ (\Diamond^{p_1-(2n+1)-1}(ba)^n bc)^\omega, (\Diamond^{p_1-(2n+1)-1}(ca)^n cb)^\omega, (\Diamond^{p_1-(2n+1)-1}(cb)^n ca)^\omega, (\Diamond^{p_1-(2n+1)-1}(bc)^n ba)^\omega \end{pmatrix}$ of infinite length over $\Sigma = \{a,b,c\}$

  lie on a sphere $(x-k)^2 + (y-k)^2 + (z-k)^2 = \left(\sum_{i=1}^{n} \frac{2^{2i+1}}{2^{p_1}-1} + \frac{1}{2^{p_1}-1} + k\right)^2 + \left(\sum_{i=1}^{n} \frac{2^{2i}}{2^{p_1}-1} + \frac{2}{2^{p_1}-1} + k\right)^2 + k^2$ ………… (3)

  where $p_1 > 2n+2$
- The number of words of infinite length over $\Sigma = \{a,b,c\}$ lying on (3) is 12.
- Word $\begin{pmatrix} (\Diamond^{p_1-(2n+1)-1}(ab)^n ca)^\omega, (\Diamond^{p_1-(2n+1)-1}(ac)^n ba)^\omega, (\Diamond^{p_1-(2n+1)-1}(ba)^n cb)^\omega, (\Diamond^{p_1-(2n+1)-1}(bc)^n ab)^\omega, \\ (\Diamond^{p_1-(2n+1)-1}(cb)^n ac)^\omega, (\Diamond^{p_1-(2n+1)-1}(ca)^n bc)^\omega, (\Diamond^{p_1-(2n+1)-1}(ba)^n ac)^\omega, (\Diamond^{p_1-(2n+1)-1}(ca)^n ab)^\omega, \\ (\Diamond^{p_1-(2n+1)-1}(ab)^n bc)^\omega, (\Diamond^{p_1-(2n+1)-1}(cb)^n ba)^\omega, (\Diamond^{p_1-(2n+1)-1}(ac)^n cb)^\omega, (\Diamond^{p_1-(2n+1)-1}(bc)^n ca)^\omega \end{pmatrix}$ ,of infinite length over $\Sigma = \{a,b,c\}$

  lie on a sphere $(x-k)^2 + (y-k)^2 + (z-k)^2 = \left(\sum_{i=1}^{n} \frac{2^{2i+1}}{2^{p_1}-1} + \frac{1}{2^{p_1}-1} + k\right)^2 + \left(\sum_{i=1}^{n} \frac{2^{2i}}{2^{p_1}-1} + k\right)^2 + \left(\frac{2}{2^{p_1}-1} + k\right)^2$ ……….(4)

  where $p_1 > 2n+2$
- The number of words of infinite length over $\Sigma = \{a,b,c\}$ lying on (4) is 12.
- Word $\begin{pmatrix} (\Diamond^{p_2-1} a \Diamond^{p_1-p_2-1} a)^\omega, (\Diamond^{p_2-1} b \Diamond^{p_1-p_2-1} b)^\omega, (\Diamond^{p_2-1} c \Diamond^{p_1-p_2-1} c)^\omega, (\Diamond^{p_2-1} a \Diamond^{p_3-p_2-1} c \Diamond^{p_1-p_3-1} b)^\omega, \\ (\Diamond^{p_2-1} a \Diamond^{p_3-p_2-1} b \Diamond^{p_1-p_3-1} c)^\omega, (\Diamond^{p_2-1} b \Diamond^{p_3-p_2-1} c \Diamond^{p_1-p_3-1} a)^\omega, (\Diamond^{p_2-1} b \Diamond^{p_3-p_2-1} a \Diamond^{p_1-p_3-1} c)^\omega, \\ (\Diamond^{p_2-1} c \Diamond^{p_3-p_2-1} a \Diamond^{p_1-p_3-1} b)^\omega, (\Diamond^{p_2-1} c \Diamond^{p_3-p_2-1} b \Diamond^{p_1-p_3-1} a)^\omega \end{pmatrix}$ where $p_3 = p_1 - p_2$ of infinite

  length over $\Sigma = \{a,b,c\}$ lie on a sphere for $n, P_1, P_2 \in N$, $x^2 + y^2 + z^2 = \left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$ where $P_1 > P_2, P_1 - P_2 \geq 2n+1$ or on a sphere

  $x^2 + y^2 + z^2 = \left(\frac{2^{p_2-p_1}+1}{2^{p_2}-1}\right)^2$ where $P_2 > P_1, P_2 - P_1 \geq 2n+1$
- The number of words of length $p_2$ over $\Sigma = \{a,b,c\}$ lying on the sphere for $n, P_1, P_2 \in N$, $x^2 + y^2 + z^2 = \left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$ where

  $P_1 > P_2, P_1 - P_2 \geq 2n+1$ or on a sphere $x^2 + y^2 + z^2 = \left(\frac{2^{p_2-p_1}+1}{2^{p_2}-1}\right)^2$ where $P_2 > P_1, P_2 - P_1 \geq 2n+1$ is 9.

### *A. Theorem 5.1*

PL is a $\omega$-Quasi-Partial spherical language of a $\omega$- partial quasi language spherical $s:(x-k_1)^2 + (y-k_2)^2 + (z-k_3)^2 = r^2$ Iff for $k_1, k_2, k_3 = k$ and $r^2 = \left(\sum_{i=1}^{n} \frac{2^{2i+1}}{2^{p_1}-1} + \frac{1}{2^{p_1}-1} + k\right)^2 + \left(\sum_{i=1}^{n} \frac{2^{2i}}{2^{p_1}-1} + k\right)^2 + \left(\frac{2}{2^{p_1}-1} + k\right)^2$ or

$\left(\sum_{i=1}^{n} \frac{2^{2i+1}}{2^{P_1}-1} + \frac{1}{2^{P_1}-1} + k\right)^2 + \left(\sum_{i=1}^{n} \frac{2^{2i}}{2^{P_1}-1} + \frac{2}{2^{P_1}-1} + k\right)^2 + k^2$ where $p_1 > 2n+2$ and for $k_1, k_2, k_3 = 0, n, P_1, P_2 \in N$ either $r^2 = \left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$ where $P_1 > P_2, P_1 - P_2 \geq 2n+1$ or $= \left(\frac{2^{p_2-p_1}+1}{2^{p_2}-1}\right)^2$ where $P_2 > P_1, P_2 - P_1 \geq 2n+1$.

Proof:

Let PL is a $\omega$-Quasi-Partial spherical language of a $\omega$-quasi-partial language spherical $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$

(i.e.) PL = { $x \in \Sigma^{\infty}$: p(PL) lies on the non-planar surface $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$ , where x is a partial words}

By theorem 3.4, centre $(k_1,k_2,k_3)=(k,k,k)$

Since $A=\left(\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ba\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n ac\right)^{\omega}\right)$

$B=\left(\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n cb\right)^{\omega}\right)$ and

$C=\left(\left(\Diamond^{p_2-1}a\Diamond^{p_1-p_2-1}a\right)^{\omega},\left(\Diamond^{p_2-1}b\Diamond^{p_1-p_2-1}b\right)^{\omega},\left(\Diamond^{p_2-1}c\Diamond^{p_1-p_2-1}c\right)^{\omega},\left(\Diamond^{p_2-1}a\Diamond^{p_3-p_2-1}c\Diamond^{p_1-p_3-1}b\right)^{\omega}\right)$ where $p_3 = p_1 - p_2$ are four non planar points. A and B lies on the non-planar surface

$s_1:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r_1^2, s_2:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r_2^2$ and $s_3:(x)^2+(y)^2+(z)^2=r_3^2$ Where

$r_1^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{p_1}-1}+\frac{1}{2^{p_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{p_1}-1}+k\right)^2+\left(\frac{2}{2^{p_1}-1}+k\right)^2$ , $r_2^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{1}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+k^2$ , $r_3^2=\left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$. If $p_1 \le 2n+2$ , contradict the PL hence $p_1 > 2n+2$

If $P_2-P_1<2n+1$, then it will contradict to the definition of GPV's hence $P_2-P_1\ge 2n+1$.

Conversely, if the centre of the sphere s is $(k_1,k_2,k_3)=(k,k,k)$ then $r^2$ is either

$\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{p_1}-1}+\frac{1}{2^{p_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{p_1}-1}+k\right)^2+\left(\frac{2}{2^{p_1}-1}+k\right)^2$ or $\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{1}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+k^2$ where $p_1 > 2n+2$

If the centre of the sphere s is $(k_1,k_2,k_3)=(0,0,0)$ then $r^2=\left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$ where either $P_1>P_2, P_1-P_2\ge 2n+1$

Since $s:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=r^2$

If $s_1:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{p_1}-1}+\frac{1}{2^{p_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{p_1}-1}+k\right)^2+\left(\frac{2}{2^{p_1}-1}+k\right)^2$ then

$A=\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ba\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n cb\right)^{\omega}$ are planar points lies in $S_1$ .

Since $\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{1}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+k^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+k\right)^2+\left(\frac{1}{2^{P_1}-1}+k\right)^2$ then

$$B=\begin{pmatrix}\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ba\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n cb\right)^{\omega},\\ \left(\Diamond^{p_1-(2n+1)-1}(ab)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n cb\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n ba\right)^{\omega}\end{pmatrix}$$

are non-planar points lies in $S_1$.

If $s_2:(x-k_1)^2+(y-k_2)^2+(z-k_3)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{1}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+k^2$ and

$C=\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n cb\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n ba\right)^{\omega}$ are planar points lies in $S_2$ .

Since $\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{p_1}-1}+\frac{1}{2^{p_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{p_1}-1}+k\right)^2+\left(\frac{2}{2^{p_1}-1}+k\right)^2=\left(\sum_{i=1}^{n}\frac{2^{2i+1}}{2^{P_1}-1}+\frac{2}{2^{P_1}-1}+k\right)^2+\left(\sum_{i=1}^{n}\frac{2^{2i}}{2^{P_1}-1}+k\right)^2+\left(\frac{1}{2^{P_1}-1}+k\right)^2$

then $$D=\begin{pmatrix}\left(\Diamond^{p_1-(2n+1)-1}(ab)^n ba\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n cb,\right)^{\omega}\\ \left(\Diamond^{p_1-(2n+1)-1}(ab)^n ac\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ac)^n ab\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ba)^n bc\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(ca)^n cb\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(cb)^n ca\right)^{\omega},\left(\Diamond^{p_1-(2n+1)-1}(bc)^n ba\right)^{\omega}\end{pmatrix}$$ are non-planar points lies in $S_2$.

If $s_3 : (x)^2 + (y)^2 + (z)^2 = \left(\frac{2^{p_1-p_2}+1}{2^{p_1}-1}\right)^2$, $E = \left(\Diamond^{p_2-1} a \Diamond^{p_1-p_2-1} a\right)^{\omega}, \left(\Diamond^{p_2-1} b \Diamond^{p_1-p_2-1} b\right)^{\omega}, \left(\Diamond^{p_2-1} c \Diamond^{p_1-p_2-1} c\right)^{\omega}, \left(\Diamond^{p_2-1} a \Diamond^{p_3-p_2-1} c \Diamond^{p_1-p_3-1} b\right)^{\omega}$ are non-planar points lies in $s_3$. If PL is a Partial language consist of words of B or D or E then PL is a $\omega$-partial-quasi spherical language of a $\omega$-partial quasi language spherical S. Hence the theorem.

*B. Example 5.1.1*

Le $PL = \{(\diamond\, ababba)^{\omega}, (\diamond\, acacca)^{\omega}, (\diamond\, babaab)^{\omega}, (\diamond\, cacaac)^{\omega}, (\diamond\, cbcbbc)^{\omega}, (\diamond\, bcbccb)^{\omega}, (\diamond\, ababac)^{\omega}, (\diamond\, acacab)^{\omega}$ $(\diamond\, bababc)^{\omega}, (\diamond\, cacacb)^{\omega}, (\diamond\, cbcbca)^{\omega}, (\diamond\, bcbcba)^{\omega}\}$. PL is a $\omega$- quasi partial spherical language and its corresponding $\omega$- quasi partial language spherical is $x^2 + y^2 + z^2 = \frac{2122}{(2^7-1)^2}$

## VI. Conclusion

In this paper, we have introduced the concept of spherical language, Language spherical, Quasi-spherical language and Quasi-language spherical. The Gpv's of all words of Quasi partial Spherical language lies within the region $x^2 + y^2 + z^2 = \frac{81}{2^8}$ and the Gpv's of all partial words of $\omega$ -quasi partial Spherical language lies within the region bounded by a surface $x^2 + y^2 + z^2 = \frac{81}{225}$

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